Functional Spaces and PDE: when Lévy, Besov, Morrey and Campanato meet

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The **Laplacian**: between analysis, P.D.E. and probability

$$\Delta(f) = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f \qquad (f : \mathbb{R}^n \longrightarrow \mathbb{R})$$

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$$\Delta(f) = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f \qquad (f : \mathbb{R}^n \longrightarrow \mathbb{R})$$

- Using Fourier $\widehat{\Delta(f)} = -c|\xi|^2 \widehat{f}$

- Heat Equation

$$\partial_t u(t,x) - \Delta u(t,x) = 0,$$

fundamental solution gaussian:

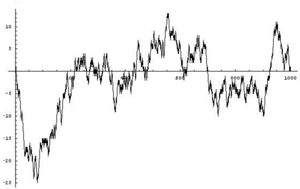
$$g_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \text{ if } t > 0.$$

Brownian motion

- If $X \sim \mathcal{N}(\sigma, m)$ then

$$\mathbb{P}(X \leq \alpha) = \int_{-\infty}^{\alpha} g_\sigma(x - m)dx$$

- Brownian motion is a special random walk



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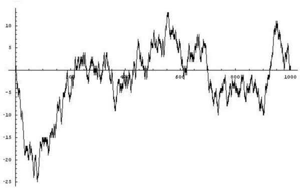

$\Longrightarrow$ The Laplacian is the *infinitesimal generator* of the Brownian motion.

Fractional Laplacian?

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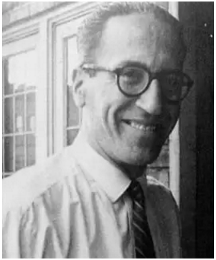
$$\left(\sqrt{(-\Delta)}(f)\right)^{\wedge} = c|\xi|\widehat{f}$$

- P.D.E.:

$$\partial_t u(t,x) + \sqrt{(-\Delta)}u(t,x) = 0$$

fundamental solution Poisson's kernel

$$p_t(x) = ct(t^2 + |x|^2)^{-\frac{n+1}{2}}$$



A. Calderón (1920-1998)

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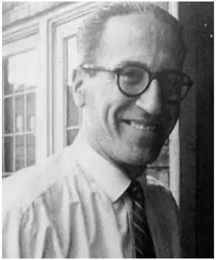

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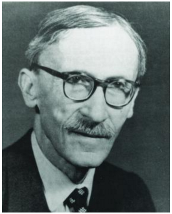
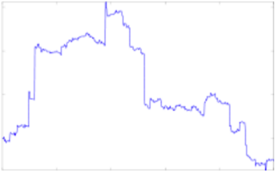
What is the corresponding stochastic process?

Lévy processes



P. Lévy (1886-1971)



- If $0 < \alpha < 2$ the operators $(-\Delta)^{\frac{\alpha}{2}}$ are the infinitesimal generators of Lévy processes (α-stables).

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it is a generalization of the fractional Laplacian $(0 < \alpha < 2)$

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Definition (Lévy-type operator)

The operator $\mathcal{L}^{\alpha}$ is given by

$$\boxed{\mathcal{L}^{\alpha}(f)(x) = v.p. \int_{\mathbb{R}^n} (f(x) - f(y))\pi(x - y)dy}$$

where $\pi(y) \sim |y|^{-(n+\alpha)}$ if $0 < |y| < 1$ and $\pi(y) \sim |y|^{-(n+\delta)}$ if $|y| > 1$.

$\Longrightarrow$ Similar regularity properties to a factional Laplacian.

Second ingredient: Functional spaces

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Functional spaces (I) - Morrey-Campanato spaces

Generalization of **Lebesgue** and **Hölder** spaces

If

$$\overline{f}_{B(x_0,r)} = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(y)dy$$

is the average of f over $B(x_0, r)$

Ch. Morrey (1907-1984)

S. Campanato (1930-2005)

Define $M^{q,a}(\mathbb{R}^n)$ with $1 \leq q < +\infty$, $0 \leq a < n+q$ by

$$\|f\|_{M^{q,a}} = \sup_{\substack{x_0\in\mathbb{R}^n\\ 0<r<1}} \left(\frac{1}{r^a}\int_{B(x_0,r)} |f(x) - \overline{f}_{B(x_0,r)}|^q dx\right)^{1/q} + \sup_{\substack{x_0\in\mathbb{R}^n\\ r\geq 1}} \left(\frac{1}{r^a}\int_{B(x_0,r)} |f(x)|^q dx\right)^{1/q} < +\infty$$

Properties (Morrey-Campanato)

- If $0 \leq a < n$ we obtain Morrey spaces

$$L^p(\mathbb{R}^n) \subset M^{q,a}(\mathbb{R}^n) \qquad \text{with } p = \frac{qn}{n-a}.$$

- If $a = n$ we obtain $bmo(\mathbb{R}^n)$
- If $n < a < n + q$ then $M^{q,a}(\mathbb{R}^n) \simeq$ Hölder $\mathcal{C}^{\lambda}(\mathbb{R}^n)$ with $0 < \frac{a-n}{q} = \lambda < 1$.

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$\Longrightarrow$ Morrey-Campanato describe singular or regular properties following the values of a

$$\begin{array}{ccccc} M^{q,a} & \longrightarrow & bmo & \longrightarrow & \mathcal{C}^\lambda \\ 0 \leq a < n & \longrightarrow & a = n & \longrightarrow & n < a < n+q \end{array}$$

Functional spaces (II) - Besov spaces

These spaces are generalization of **Sobolev** spaces.

O. Besov (1933-)

- If $0 < s < 2$, $1 \leq p, q \leq +\infty$

$$\|f\|_{\dot{B}^{s,p}_q} = \left(\int_{\mathbb{R}^n} \frac{\|f(\cdot + y) + f(\cdot - y) - 2f(\cdot)\|^q_{L^p}}{|y|^{n+sq}} dy\right)^{1/q} < +\infty.$$

- If $0 < s < 1$, $p = q$ we have:

$$\|f\|_{\dot{B}^{s,p}_p} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dxdy\right)^{1/p} < +\infty.$$

$\Longrightarrow$ measuring the regularity of functions.

Properties (Besov)

- If $0 < s < 1$ and $p = q = +\infty$ we have $\dot{B}^{s,\infty}_{\infty} = \dot{\mathcal{C}}^{s}$ (Hölder spaces)
- We have $\|(-\Delta)^{\frac{\alpha}{2}} f\|_{\dot{B}^{s-\alpha,p}_{q}} = \|f\|_{\dot{B}^{s,p}_{q}}$

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Theorem (D.Ch. & PG. Lemarié ('12) + D.Ch & S. Menozzi ('15))

- *Let $\mathcal{L}^\alpha$ be a Lévy-type operator of regularity $0 < \alpha < 2$.*
- *If $2 \leq p < +\infty$, we have the inequality*

$$\|f\|_{\dot{B}_p^{\frac{\alpha}{p},p}} \leq \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \mathcal{L}^\alpha f(x) dx$$

$\Longrightarrow$ First meeting between Besov and Lévy.

The Equation

Transport-Diffusion

- Consider $\theta : [0, +\infty[\times \mathbb{R}^n \longrightarrow \mathbb{R}$ with $n \geq 2$
- Let $\mathcal{L}^\alpha$ be a Lévy-type operator with regularity $0 < \alpha < 2$ (Diffusion).
- Let $A_{[\theta]}(t,x) = [A_1(\theta)(t,x), \cdots, A_n(\theta)(t,x)]$ be a singular integral vector field (Transport) *i.e.*

$$A_i(\theta)(t,x) = v.p. \int_{\mathbb{R}^n} \kappa_i(x-y)\theta(t,y)dy$$

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$$\begin{cases} \partial_t\theta(t,x) - \nabla\cdot(A_{[\theta]}\,\theta)(t,x) + \mathcal{L}^\alpha\theta(t,x) = 0, \\ div(A_{[\theta]}) = 0, \\ \theta(0,x) = \theta_0(x), \quad \textit{for all } x \in \mathbb{R}^n. \end{cases}$$

$\Longrightarrow$ We have a non-linear transport-diffusion equation

Aim of the work

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We want to study the **regularity** of the solutions of this equation.

Some properties of the equation

(i) If $\theta_0 \in L^p(\mathbb{R}^n)$ with $1 < p < +\infty$

(ii) If the drift is bounded in Morrey spaces

$$\|A_{[\theta]}\|_{L^\infty(M^{q,a})} \leq C\|\theta\|_{L^\infty(M^{q,a})}$$

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$\Longrightarrow$ existence of weak solutions $\theta \in L^\infty(L^p)$

$\Longrightarrow$ we have a maximum principle

$$\boxed{\|\theta(t,\cdot)\|_{L^p} \leq \|\theta_0\|_{L^p}}$$

Theorem (D. Ch. & S. Menozzi ('15))

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$$\mathcal{C}^{1-\alpha}$$

$$\uparrow$$

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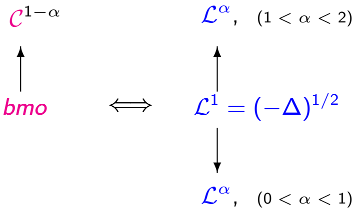

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$$\begin{array}{ccc}
\mathcal{C}^{1-\alpha} & & \mathcal{L}^{\alpha},\quad (1<\alpha<2) \\
\uparrow & & \uparrow \\
bmo & \Longleftrightarrow & \mathcal{L}^{1} = (-\Delta)^{1/2} \\
\downarrow & & \downarrow \\
\mathcal{M}^{q,a} & & \mathcal{L}^{\alpha},\quad (0<\alpha<1)
\end{array}$$

Can we break this relationship?

$$\frac{a-n}{q} = 1 - \alpha$$

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$\Longrightarrow$ In the general case (linear drift) **no**

$\Longrightarrow$ In a **non-linear** case it is possible. . .

Main Idea: use additional information $A_{[\theta]} = f(\theta)$

Properties

A further study of the maximum principle gives

$$\|\theta(t,\cdot)\|_{L^p} + \int_0^t \underbrace{\int_{\mathbb{R}^n} |\theta|^{p-2}\theta\mathcal{L}^\alpha\theta dx}_{\|\theta\|_{\dot{B}_p^{\frac{\alpha}{p},p}} \leq} ds \leq \|\theta_0\|_{L^p} < +\infty$$

$\Longrightarrow$ We have a control of the quantity $\|\theta\|_{\dot{B}_p^{\frac{\alpha}{p},p}}$: the solution belongs to a Besov space.

$\Longrightarrow$ We will use this information!

Theorems

Theorem (1)

$$\partial_t \theta(t,x) - \nabla \cdot (A_{[\theta]}\, \theta)(t,x) + \mathcal{L}^{\alpha_0} \theta(t,x) = 0 \qquad \theta_0 \in L^p$$

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$$\|A_{[\theta]}\|_{\dot{B}_p^{\frac{\alpha_0}{p},p}} \leq C\|\theta\|_{\dot{B}_p^{\frac{\alpha_0}{p},p}}$$

(iv) *If* $p = \frac{qn}{n-a}$, *and* $1 < \alpha_0 = \frac{p+n}{p+1} < 2$,

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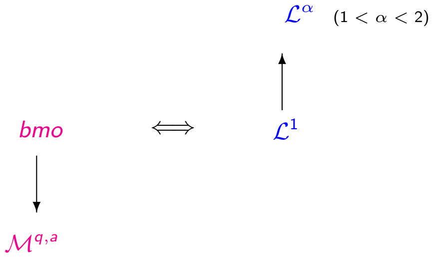
The Besov condition is quite *general*

$\Longrightarrow$ First meeting between Lévy, Besov and Morrey-Campanato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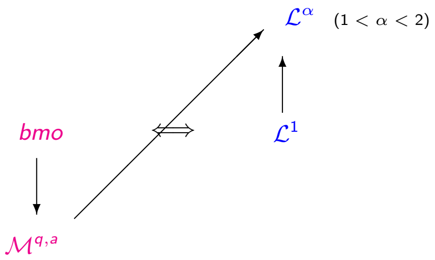
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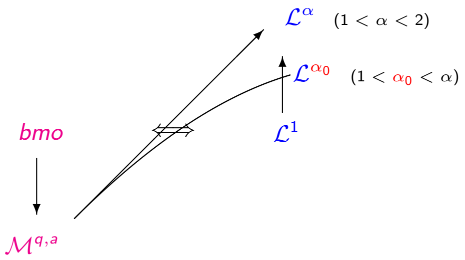

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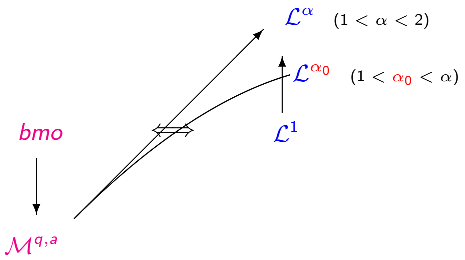

The Besov information allows to break the equilibrium

always useful, but if $a \to n$ ($\mathcal{M}^{q,a} \to bmo$)

then we have $p \to +\infty$ and $\dot{B}_p^{\frac{\alpha_0}{p},p} \to \dot{B}_\infty^{0,\infty}$.

Theorem (2)

$$\partial_t \theta(t,x) - \nabla \cdot (A_{[\theta]}\, \theta)(t,x) + \mathcal{L}^{\alpha_0}\theta(t,x) = 0 \qquad \theta_0 \in L^p$$

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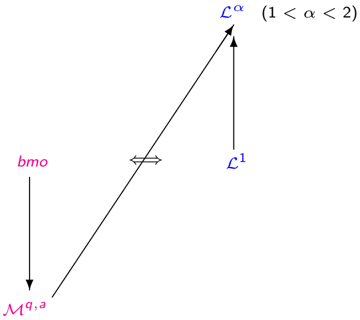
$\Longrightarrow$ **Competition** between Lévy, Besov and Morrey-Campanato



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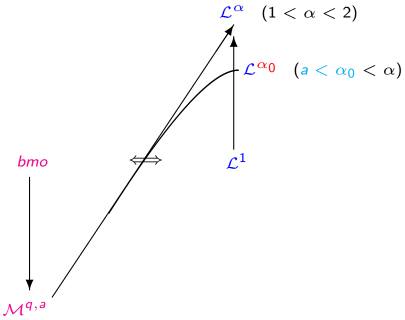
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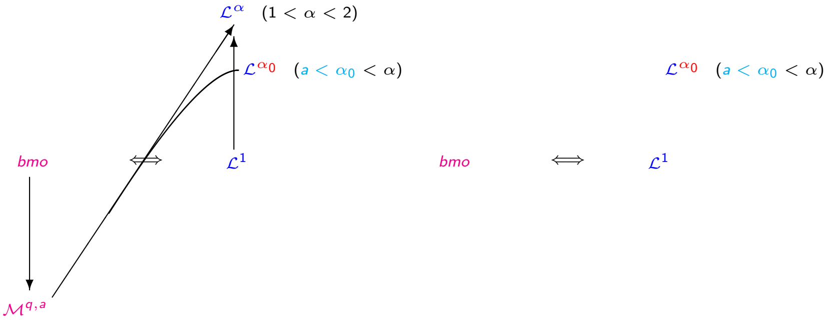
$\mathcal{L}^{\alpha}$ $(1 < \alpha < 2)$
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bmo $\Longleftrightarrow$ $\mathcal{L}^1$
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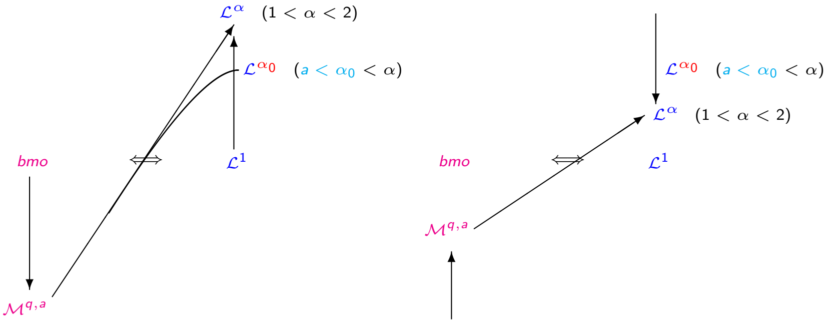
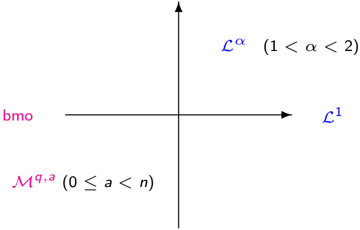


As Morrey-Campanato/Lebesgue spaces become less singular the Besov information becomes useless


$\mathcal{L}^{\alpha}$ $(1 < \alpha < 2)$
bmo
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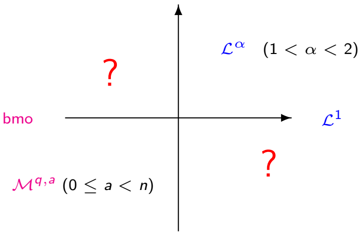
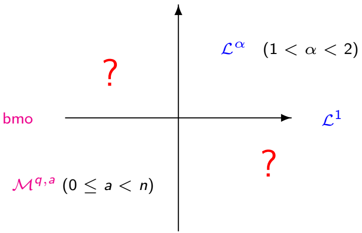


It is possible to the break the equilibrium when $0 < \alpha \leq 1$?



It is possible to the break the equilibrium when $0 < \alpha \leq 1$?

open problem!

Theorem (3)

$$\partial_t\theta(t,x) - \nabla\cdot(A_{[(-\Delta)^{-\varepsilon/2}\theta]}\,\theta)(t,x) + \mathcal{L}^{\alpha_0}\theta(t,x) = 0 \qquad \theta_0 \in L^p \qquad 0 < \varepsilon < 1$$

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(iv) *if* $p = \frac{qn}{q\varepsilon+n-a}$, *and* $1/2 < \alpha = \frac{n+p(1-\varepsilon)}{p+1} < 2.$

then the solutions are Hölder regular.

$\Longrightarrow$ The **Smoothing effect** of $(-\Delta)^{-\varepsilon/2}$ allows $1/2 < \alpha < 1$.

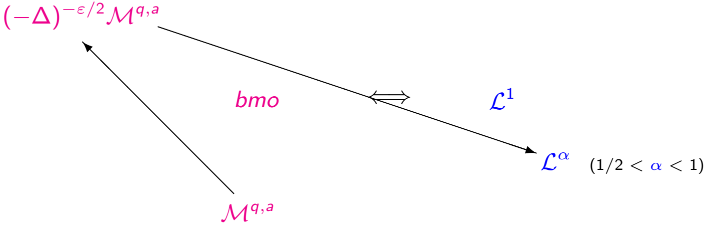
$$bmo \quad \Longleftrightarrow \quad \mathcal{L}^1$$

$$\mathcal{M}^{q,a}$$

$(-\Delta)^{-\varepsilon/2}\mathcal{M}^{q,a}$

$bmo \iff \mathcal{L}^1$

$\mathcal{M}^{q,a}$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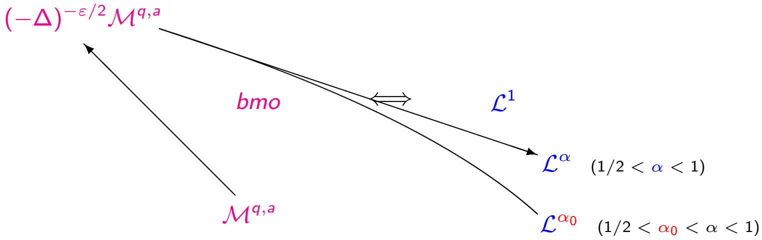
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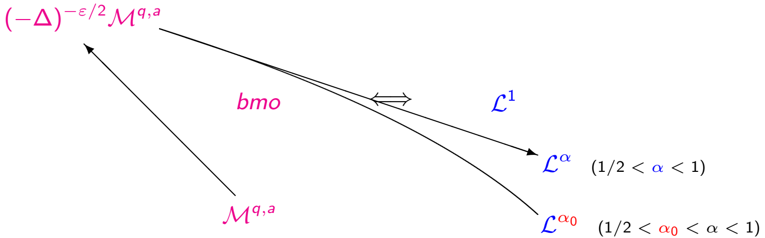

$\Longrightarrow$ **Smoothing effect** of $(-\Delta)^{-\varepsilon/2}$ + Besov information

Theorem (4)

$$\partial_t\theta(t,x) - \nabla\cdot(A_{[(-\Delta)^{+\varepsilon/2}\theta]}\,\theta)(t,x) + \mathcal{L}^{\alpha_0}\theta(t,x) = 0 \qquad \theta_0 \in L^p \qquad 0 < \varepsilon < 1$$

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$\Longrightarrow$ **Singularisation effect** with $(-\Delta)^{+\varepsilon/2}$

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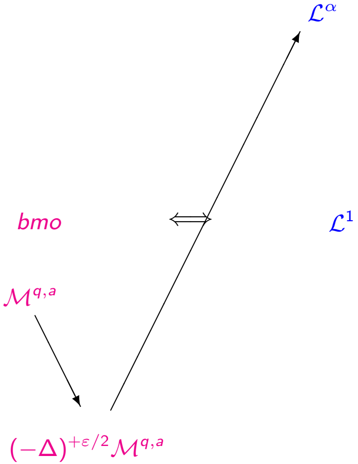
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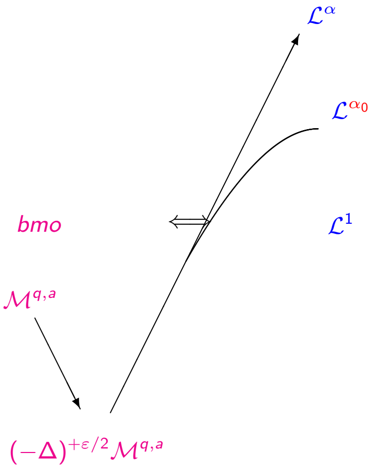
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Proof

Main Idea: duality

Definition (Hölder spaces)

If $\theta : \mathbb{R}^n \longrightarrow \mathbb{R}$, *then* $\theta \in \mathcal{C}^\gamma$ $(0 < \gamma < 1)$ *if*

$$\|\theta\|_{\mathcal{C}^\gamma} = \|\theta\|_{L^\infty} + \sup_{x \neq y} \frac{|\theta(x) - \theta(y)|}{|x-y|^\gamma} < +\infty$$

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Theorem (Hardy-Hölder duality (Goldberg '79))

If $0 < \gamma < 1$ and $0 < \sigma < 1$ are s.t. $\sigma = \frac{n}{n+\gamma}$

$\Longrightarrow$ *The dual space of a* **Hardy** *space h^σ is the* **Hölder** *space $\mathcal{C}^\gamma$.*

$\Longrightarrow$ $\theta \in \mathcal{C}^\gamma$ iff for all $g \in h^\sigma$ we have $\langle \theta, g \rangle_{\mathcal{C}^\gamma \times h^\sigma} < +\infty$

Molecular Hardy spaces

Definition (Hardy spaces)

If $\mathbf{g} \in h^\sigma$ *then*

$$\mathbf{g} = \sum_{j \in \mathbb{N}} \lambda_j \psi_j$$

where $\sum_{j\in\mathbb{N}} |\lambda_j|^\sigma < +\infty$ *and* $(\psi_j)_{j\in\mathbb{N}}$ *are r-molecules.*

Definition (γ-Molecules)

- $(0 < r < 1)$:

$$\int_{\mathbb{R}^n} |\psi(x)||x - x_0|^\omega dx \quad \leq \quad r^{\omega-\gamma}, \textit{ with } x_0 \in \mathbb{R}^n$$

$$\|\psi\|_{L^\infty} \leq \frac{1}{r^{n+\gamma}} \quad \textit{and} \quad \int_{\mathbb{R}^n} \psi(x)dx = 0$$

- $(1 \leq r < +\infty)$: *same conditions, without cancellation condition*

$\Longrightarrow$ these conditions imply $(1 \leq p < +\infty)$

$$\|\psi\|_{L^p} \leq \frac{1}{r^{(n-\frac{n}{p}+\gamma)}}, \qquad \text{in particular } \|\psi\|_{L^1} \leq \frac{1}{r^\gamma}$$

Trick: Transfert

Theorem (Kiselev & Nazarov ('10))

$\Longrightarrow$ *If* θ *sol. Eq.* $\partial_t\theta + \nabla\cdot(v\theta) + \mathcal{L}(\theta) = 0$

$\Longrightarrow$ *If* ψ *sol. Eq. with* ψ_0 *a molecule.*

$$\partial_t\psi + \nabla\cdot(v\psi) + \mathcal{L}(\psi) = 0$$

then there is a **conservation** *of the duality bracket.*

$$\begin{array}{c}\langle\theta_0, \psi(t,\cdot)\rangle \\ \downarrow \quad \uparrow \\ \langle\theta(t,\cdot), \psi_0\rangle\end{array}$$

$\Longrightarrow$ Thus, in order to study the regularity of the solutions...

$$|\langle \theta(t,\cdot), \psi_0 \rangle| = |\langle \theta_0, \psi(t,\cdot) \rangle| \leq \|\theta_0\|_{L^\infty} \|\psi(t,\cdot)\|_{L^1}$$

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Theorem

For all molecular initial data ψ_0, the solution $\psi(t,x)$ is controlled in L^1-norm for $t > T_0$.

- for big molecules $\Longrightarrow$ maximum principle

$$\|\psi\|_{L^1} \leq \frac{1}{r^\gamma}$$

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- we need to study small molecules.

Molecules Evolution (I)

- For a small r-molecule:

$$\int |\psi(0,x)||x-x_0|^{\omega}dx \leq r^{\omega-\gamma}$$

$$\|\psi(0,\cdot)\|_{L^\infty} \leq \frac{1}{r^{n+\gamma}}$$

- For a small time s we want to obtain a molecule of size $(r+s)$:

$$\int |\psi(s,x)||x-x_s|^{\omega}dx \leq (r^{\alpha}+Ks)^{(\omega-\gamma)/\alpha}$$

$$\|\psi(s,\cdot)\|_{L^\infty} \leq \frac{1}{(r^{\alpha}+Ks)^{(n+\gamma)/\alpha}}$$

Hint: the drift is singular, the displacement of the molecule's center x_0 will be given by an average.

$$\begin{cases} x'(s) = & \overline{v}_{B(x(s),r)} = \frac{1}{|B(x(s),r)|}\displaystyle\int_{B(x(s),r)} v(s_0, y)dy, \quad s \in [0, s_0], \\ x(0) = & x_0. \end{cases}$$

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$\Longrightarrow$ Morrey-Campanato will appear with this average.

But this is *not* the only way to "average"...

$$x'(s) = \overline{v}_{B(x(s),r)} = v*\varphi, \quad \varphi\in\mathcal{C}_0^\infty$$

$\Longrightarrow$ Besov spaces will appear from this mollification.

Molecules Evolution (II)

We "**melt**" the L^1-norm of molecules:

$$\int_{\mathbb{R}^n} |\psi(S_0, x)||x - x_{S_0}|^{\omega} dx \leq C_1 \quad + \quad \|\psi(S_0, \cdot)\|_{L^\infty} \leq C_2$$

$\Longrightarrow$ $\|\psi(S_0, \cdot)\|_{L^1} \leq C_3$

$\Longrightarrow$ We then obtain a suitable L^1-norm control. ■

$\Longrightarrow$ Robust method:

- tools from harmonic analysis
- some generalizations are possible (Stratified Lie groups)
- more general Lévy-type operators ($\pi(x, y, t)$)

$\Longrightarrow$ However this method depend on

- $div(v) = 0$
- dual equation

$\Longrightarrow$ open problem: break the equilibrium when $0 < \alpha < 1$ (??)

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