Smoothing properties of Kolmogorov Equations and Regularization by noise: degenerate and McKean-Vlasov cases

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REGULARISATION BY NOISE

and

SMOOTHING PROPERTIES OF KOLMOGOROV EQUATION

A PRIMER

Well-posedness of ordinary differential equation

Consider on $[0, T]$, $T > 0$ the system :

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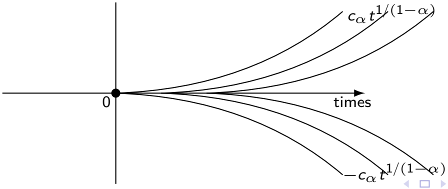

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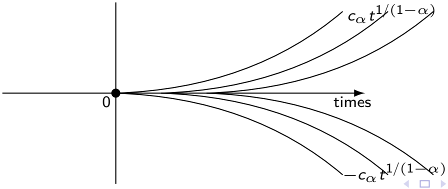

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Infinitesimal perturbation regularizes the dynamic

↳ REGULARIZATION BY NOISE PHENOMENON

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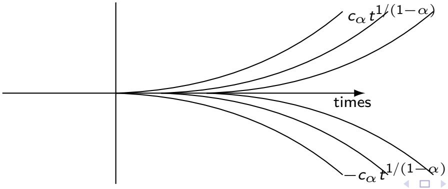
$$X_t = \pm c_\alpha (t - t^\star)^{1/(1-\alpha)} \mathbf{1}_{[t^\star, +\infty)}(t), \ \ t^\star \in [0, T].$$



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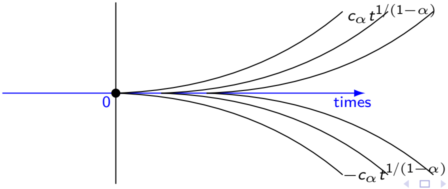

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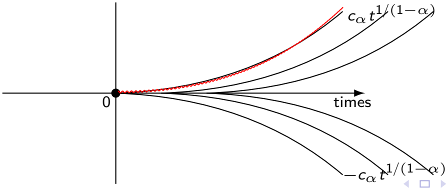

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  ↳ It must exists $t_\Sigma < 1$ such that
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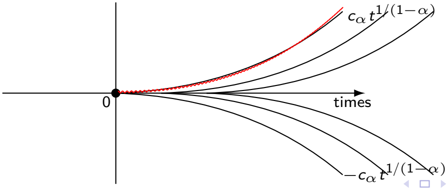

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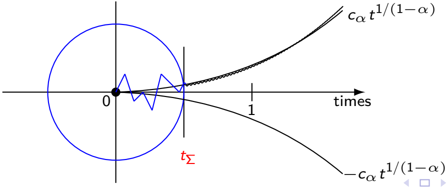

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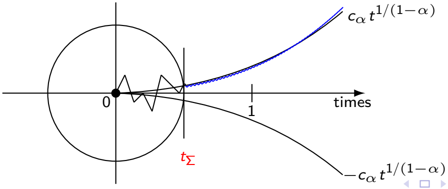

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↳ The noise does not degenerate → the process X visits all the space around the initial condition → mix the values of solution of Cauchy problem associated to $\partial_t + \mathcal{L}$ → regularizes it

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$\Longrightarrow$ Existence and uniqueness on small time intervals

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$\Longrightarrow$ Existence and uniqueness on small time intervals $\xrightarrow{\text{“Markov“}}$ $\exists!\ \mathbb{R}^+$

Regularization by noise, a PDE point of view

Consider on $[0, T]$, $T > 0$ the stochastic system :

$$X_0 = x, \quad \mathrm{d}X_t = F(X_t)\mathrm{d}t + \mathrm{d}B_t$$

where $(B_t)_{t\geqslant 0}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

- Why the infinitesimal perturbation regularizes the differential system ?
  ↳ Zvonkin's transformation (aka Itô-Tanaka "trick") :
  ↳ F writes (Itô Formula on $u(t, X_t)$) :

$$\int_0^t F(X_s)\mathrm{d}s = u(t, X_t) - u(0, x) - \int_0^t \partial_x u(s, X_s)\mathrm{d}B_s$$

↳ where u is the solution of

$$\partial_t u(t,x) + \mathcal{L}u(t,x) = F(x) \text{ on } [0,T) \times \mathbb{R}, \quad u(T,x) = 0, \qquad \mathcal{L} \text{ generator of } X$$

- If F is α-Hölder, $\alpha \in (0,1) \to$ u and $\partial_x u$ are $(||u_T||_{\mathrm{Lip}} + C_T)$ resp. $(||\partial_x u_T||_{\mathrm{Lip}} + C'_T)$ Lipschitz
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$\Longrightarrow$ Existence and uniqueness on small time intervals $\overset{\text{"Markov"}}{\longrightarrow}$ $\exists! \ \mathbb{R}^+$

Existence and uniqueness on small time intervals : proof

Consider on $[0, T]$, $T > 0$ the stochastic system :

$$X_0 = x_0, \quad \mathrm{d}X_t = F(X_t)\mathrm{d}t + \mathrm{d}B_t$$

- Suppose $(X_t, t \geqslant 0)$ and $(Y_t, t \geqslant 0)$ satisfy the SDE with initial condition x_0
- Zvonkin (or Itô-Tanaka trick) : Itô on $X_t - u(t, X_t)$
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- $$X_t - u(t, X_t) = x_0 - u(0, x_0) - \int_0^t \partial_x u(s, X_s)\mathrm{d}B_s + \int_0^t F(X_s)\mathrm{d}s - \int_0^t [\partial_t u(s, X_s) + \mathcal{L}u(s, X_s)]\,\mathrm{d}s$$

Existence and uniqueness on small time intervals : proof

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- $$X_t - u(t, X_t) = x_0 - u(0, x_0) - \int_0^t \partial_x u(s, X_s)\mathrm{d}B_s + \int_0^t F(X_s)\mathrm{d}s - \underbrace{\int_0^t [\partial_t u(s, X_s) + \mathcal{L}u(s, X_s)]\,\mathrm{d}s}_{=\int_0^t F(X_s)\mathrm{d}s}$$

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- u solution of the PDE $(\partial_t + \mathcal{L})u = F, \;\; u_T = 0$
- $X_t - u(t, X_t) = x_0 - u(0, x_0) - \int_0^t \partial_x u(s, X_s)\mathrm{d}B_s$
- $Y_t - u(t, Y_t) = x_0 - u(0, x_0) - \int_0^t \partial_x u(s, Y_s)\mathrm{d}B_s$

Existence and uniqueness on small time intervals : proof

Consider on $[0, T]$, $T > 0$ the stochastic system :

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- u solution of the PDE $(\partial_t + \mathcal{L})u = F, \; u_T = 0$
- $X_t - u(t, X_t) = x_0 - u(0, x_0) - \int_0^t \partial_x u(s, X_s)\mathrm{d}B_s$
- $Y_t - u(t, Y_t) = x_0 - u(0, x_0) - \int_0^t \partial_x u(s, Y_s)\mathrm{d}B_s$
- $\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |X_t - Y_t|^2\right] \leqslant 2\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |u(t, X_t) - u(t, Y_t)|^2\right] + 2\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} \left|\int_0^t \partial_x u(s, Y_s) - \partial_x u(s, X_s)\mathrm{d}B_s\right|^2\right]$

Existence and uniqueness on small time intervals : proof

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- $$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |X_t - Y_t|^2\Big] \leqslant 2\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |u(t, X_t) - u(t, Y_t)|^2\Big] + \mathbb{E}\int_0^T |\partial_x u(s, X_s) - \partial_x u(s, Y_s)|^2 \,\mathrm{d}s\Big\}$$

- Doob maximal inequality

Existence and uniqueness on small time intervals : proof

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- Doob maximal inequality
- u and $\partial_x u$ are $C(T)$ and $C'(T)$ Lipschitz

Existence and uniqueness on small time intervals : proof

Consider on $[0, T]$, $T > 0$ the stochastic system :

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- u and $\partial_x u$ are $C(T)$ and $C'(T)$ Lipschitz and $C(T), C'(T) \to 0$ when $T \to 0$

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- T "sufficiently small"

Existence and uniqueness on small time intervals : proof

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- u and $\partial_x u$ are $C(T)$ and $C'(T)$ Lipschitz and $C(T), C'(T) \to 0$ when $T \to 0$
- T "sufficiently small"

Regularization by noise, conclusion of the primer : sufficient condition

Consider on $[0, T]$, $T > 0$ the stochastic system :

$$X_0 = x, \quad \mathrm{d}X_t = F(X_t)\mathrm{d}t + \mathrm{d}B_t$$

where $(B_t)_{t\geqslant 0}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

- When is the system well-posed outside the Cauchy-Lipschitz framework ?

↳ Zvonkin's transformation (Itô-Tanaka trick) :

A sufficient condition is to obtained appropriate regularization properties of the associated Kolmogorov PDE with the drift as source term :

$$\partial_t u(t,x) + \mathcal{L}u(t,x) = F(x) \text{ on } [0, T) \times \mathbb{R}, \quad u(T,x) = 0$$

↳ Link with PDE and smoothing effect of linear partial second order differential operator with elliptic diffusion matrix.

Stochastic **infinitesimal perturbation** of the dynamic

DEGENERATE KOLMOGOROV EQUATION

and

REGULARIZATION BY STOCHASTIC DRIFT

Well-posedness of differential equation

Consider on $[0, T]$, $T > 0$ the system :

$$X_0 = x, \quad \mathrm{d}X_t = F(X_t)\mathrm{d}t$$

Well-posedness of differential equation

Consider on $[0, T]$, $T > 0$ the system :

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- Quid of a macroscopic perturbation ?

Well-posedness of differential equation with macroscopic perturbation

Consider on $[0, T]$, $T > 0$ the stochastic system :

$$X_0 = x, \quad \mathrm{d}X_t = (F(X_t) + \Gamma B_t)\mathrm{d}t$$

where $(B_t)_{t \geqslant 0}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, Γ in $\mathbb{R}$.

- Quid of a macroscopic perturbation ?

Well-posedness of differential equation : **regularization by stochastic drift**

Consider on $[0, T]$, $T > 0$ the stochastic system :

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Well-posedness of differential equation : regularization by stochastic drift

Consider on $[0, T]$, $T > 0$ the stochastic system :

$$\begin{array}{ll} X_0^1 = x_1, & \mathrm{d}X_t^1 = \mathrm{d}B_t \\ X_0^2 = x_2, & \mathrm{d}X_t^2 = (F(X_t^2) + \Gamma X_t^1)\mathrm{d}t \end{array}$$

where $(B_t)_{t \geqslant 0}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, Γ in $\mathbb{R}$.

- Quid of a macroscopic perturbation ?
- Here $\Sigma = \begin{pmatrix} \mathrm{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$: the system is said to be totally degenerate
- The noise in the second component only comes from the first component

Well-posedness of differential equation : regularization by stochastic drift

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Well-posedness of differential equation : regularization by stochastic drift

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  ↳ Zvonkin transformation ?
  ↳ no ellipticity

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where $(B_t)_{t\geqslant 0}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, Γ in $\mathbb{R}^*$.

- Quid of a macroscopic perturbation ?
- Here $\Sigma = \begin{pmatrix} \mathrm{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$: the system is said to be totally degenerate
- The noise in the second component only comes from the first component
- Well-posedness outside the Cauchy Lipschitz framework ?
  ↳ Zvonkin transformation ?
  ↳ no ellipticity... but hypoellipticity holds !
- Lie brackets :

$$V_0 = \begin{pmatrix} 1 \\ 0 \\ \Gamma x_1 \end{pmatrix} \quad , \quad V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad [V_0, V_1] = \begin{pmatrix} 0 \\ 0 \\ \Gamma \end{pmatrix}.$$

  $(V_0, V_1, [V_0, V_1])$ span $\mathbb{R}^3$.

Well-posedness of differential equation : regularization by stochastic drift

Consider on $[0, T]$, $T > 0$ the stochastic system :

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where $(B_t)_{t \geqslant 0}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, Γ in $\mathbb{R}$.

- Quid of a macroscopic perturbation ?

Result : If $\Gamma \neq 0$ and F is $\beta > 2/3$-Hölder *i.e.* if weak Hörmander conditions hold, then, the system is well-posed (strong sense)

Well-posedness of differential equation : regularization by stochastic drift

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Result : If $\Gamma \neq 0$ and F is $\beta > 2/3$-Hölder *i.e.* if weak Hörmander conditions hold, then, the system is well-posed (strong sense)

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- Quid of a macroscopic perturbation ?

> **Result :** If $\Gamma \neq 0$ and F is $\beta > 2/3$-Hölder *i.e.* if weak Hörmander conditions hold, then, the system is well-posed (strong sense)

- Degeneracy → loss of regularization effect of the generator $\mathcal{L}$ of (X^1, X^2)

Regularization by stochastic drift : proof. The PDE

- Strong theory for the PDE :

$$\partial_t u_i(t,x_1,x_2) + \mathcal{L}u_i(t,x_1,x_2) = \phi_i(t,x_1,x_2) \text{ on } [0,T)\times\mathbb{R}^2$$
$$u_i(T,x_1,x_2) = \mathbf{0}_{\mathbb{R}^2}, \qquad i=1,2$$

  where $\mathcal{L} := \langle F, D_{x_2}\rangle + \frac{1}{2}\Delta_{x_1}$ and $(\phi_1,\phi_2) = (\mathbf{0}, F(\cdot)+\Gamma)$.

- Is there exists a unique solution $u=(u_1,u_2)^*$ such that :

$$||u||_{\mathrm{Lip}} + ||D_{x_1}u||_{\mathrm{Lip}} \leqslant C(T),$$

  where $C(T)\to 0$ when $T\to 0$.

Regularization by stochastic drift : proof. The PDE

- Strong theory for the PDE :

$$\begin{array}{l} \partial_t u_i(t,x_1,x_2) + \mathcal{L} u_i(t,x_1,x_2) = \phi_i(t,x_1,x_2) \text{ on } [0,T) \times \mathbb{R}^2 \\ u_i(T,x_1,x_2) = \mathbf{0}_{\mathbb{R}^2}, \qquad i = 1,2 \end{array}$$

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Regularization by stochastic drift : proof. The PDE

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  - The PDE degenerates... but satisfies weak Hörmander condition.

Regularization by stochastic drift : proof. The PDE

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  - The PDE degenerates... but satisfies weak Hörmander condition.

Regularization by stochastic drift : proof. The PDE

- Strong theory for the PDE :

$$\begin{aligned} &\partial_t u_i{}^n(t,x_1,x_2) + \mathcal{L}^n u_i{}^n(t,x_1,x_2) = \phi_i{}^n(t,x_1,x_2) \text{ on } [0,T) \times \mathbb{R}^2 \\ &u_i{}^n(T,x_1,x_2) = \mathbf{0}_{\mathbb{R}^2}, \qquad i = 1,2 \end{aligned}$$

  where $\mathcal{L}^n := \langle F^n, D_{x_2}\rangle + \frac{1}{2}\Delta_{x_1}$ and $(\phi_1{}^n, \phi_2{}^n) = (\mathbf{0}, F^n(\cdot) + \Gamma)$, $n \in \mathbb{N}$.

- Is there exists a unique solution $u^n = (u_1{}^n, u_2{}^n)^*$ such that :

$$||u^n||_{\mathrm{Lip}} + ||D_{x_1} u^n||_{\mathrm{Lip}} \leqslant C(T),$$

  where $C(T) \to 0$ when $T \to 0$ and does not depend on n.

  - The PDE degenerates... but satisfies weak Hörmander condition.
  - Smooth setting : $(F^n)_{n\geqslant 0}$ sequence of regularized coefficients that satisfy the same assumptions as F (uniformly in n) and that converges uniformly to F.

Regularization by stochastic drift : proof. The PDE

- Strong theory for the PDE :

$$\partial_t {u_i}^n(t,x_1,x_2) + \mathcal{L}^n {u_i}^n(t,x_1,x_2) = {\phi_i}^n(t,x_1,x_2) \text{ on } [0,T) \times \mathbb{R}^2$$
$${u_i}^n(T,x_1,x_2) = \mathbf{0}_{\mathbb{R}^2}, \qquad i = 1,2$$

  where $\mathcal{L}^n := \langle F^n, D_{x_2} \rangle + \frac{1}{2}\Delta_{x_1}$ and $({\phi_1}^n, {\phi_2}^n) = (\mathbf{0}, F^n(\cdot) + \Gamma)$, $n \in \mathbb{N}$.

- For each n, there exists a unique solution $u^n = ({u_1}^n, {u_2}^n)^*$. Does it satisfy :

$$||u^n||_{\mathrm{Lip}} + ||D_{x_1} u^n||_{\mathrm{Lip}} \leqslant C(T),$$

  where $C(T) \to 0$ when $T \to 0$ and does not depend on n?

  - The PDE degenerates... but satisfies weak Hörmander condition.
  - Smooth setting : $(F^n)_{n \geqslant 0}$ sequence of regularized coefficients that satisfy the same assumptions as F (uniformly in n) and that converges uniformly to F.

Regularization by stochastic drift : proof. The PDE

- Strong theory for the PDE :

$$\begin{array}{l} \partial_t {u_i}^n(t,x_1,x_2) + \mathcal{L}^n {u_i}^n(t,x_1,x_2) = {\phi_i}^n(t,x_1,x_2) \text{ on } [0,T) \times \mathbb{R}^2 \\ {u_i}^n(T,x_1,x_2) = \mathbf{0}_{\mathbb{R}^2}, \qquad i = 1,2 \end{array}$$

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$$||D_{x_2} u^n||_\infty + ||D_{x_1} u^n||_\infty + ||D^2_{x_1} u^n||_\infty + ||D_{x_2} D_{x_1} u^n||_\infty \leqslant C(T),$$

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1. Representation of the solution ?

↳ Perturbed approach / Parametrix

Regularization by stochastic drift : proof. The PDE

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1. Representation of the solution ?

   ↳ Perturbed approach / Parametrix

2. Estimation of the derivatives

Regularization by stochastic drift : proof. The parametrix : degenerate case

Expand the generator around a frozen generator. In small time, both generators should be closed. The frozen generator enjoys well-known properties

- Hypoelliptic operator $\mathcal{L} := (1/2)\Delta_{x_1} + (\Gamma x_1 + F(x_2))D_{x_2}$, unbounded drift

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- PDE solution :

$$u_i(t,x_1,x_2) = \int_t^T \left[\tilde{P}^{?}_{t,s}F(\cdot) + \Gamma\right](s,x_1,x_2)\mathbf{1}_{i=2} + \left[\tilde{P}^{?}_{t,s}(\mathcal{L} - \tilde{\mathcal{L}}^{?})u\right](s,x_1,x_2)\mathrm{d}s$$

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Regularization by stochastic drift : proof. The parametrix : degenerate case

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  ↳ Frozen process : $\tilde{X}_s^{2,\theta_{t,s}(x)} = x_2 + \underbrace{\int_t^s \Gamma x_1 + F(\theta_{t,s}(x))\mathrm{d}r}_{\sim(s-t)} + \underbrace{\int_t^s \Gamma B_r \mathrm{d}r}_{\sim(s-t)^{3/2}}$

- First order parametrix : $\mathcal{L} = \tilde{\mathcal{L}}^{\theta_{t,s}(x)} + (\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(x)})$, where $\tilde{\mathcal{L}}^?$ is the generator of $\tilde{P}^{\theta_{t,s}(x)}$

- Degenerate Gaussian bound for $\tilde{P}^{\theta_{t,s}(x)}$: for all $t < s$ in $[0, T]^2$
  - $\forall \varphi \in \mathcal{B}(\mathbb{R})$, $\left| \left[\tilde{P}_{t,s}^{x}\varphi\right](s,x)\right| \leqslant C\left[\hat{P}_{t,s}^{x}|\varphi|\right](s,x)$,
  - $\hat{P}_{t,s}^{x}$: degenerate heat semi-group

- Choice of the freezing point : perturbation must have same order than typical trajectory of frozen process → freeze around θ, the transport of initial condition :

$$\dot{\theta}_{t,s}(x_1,x_2) = F(\theta_{t,s}(x_1,x_2)) + \Gamma x_1, \quad \theta_{t,t}(x_1,x_2) = x_2$$

- PDE solution :

$$u_i(t,x_1,x_2) = \int_t^T \left[\tilde{P}_{t,s}^{\theta_{t,s}(x)} F(\cdot) + \Gamma\right](s,x_1,x_2)\mathbf{1}_{i=2} + \left[\tilde{P}_{t,s}^{\theta_{t,s}(x)}(\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(x)})u\right](s,x_1,x_2)\mathrm{d}s$$

Regularization by stochastic drift : proof. Derivatives estimates

$$u_i(t, x_1, x_2)$$

- Estimation of $(D_{x_1} + D_{x_2} + D^2_{x_1} + D_{x_1}D_{x_2})u_i$

Regularization by stochastic drift : proof. Derivatives estimates

$$u_i(t, x_1, x_2)$$

- Estimation of $(D_{x_1} + D_{x_2} + D_{x_1}^2 + D_{x_1} D_{x_2}) u_i$
- Fix $\xi = (\xi_1, \xi_2) \ \rightarrow \theta$ solves $\dot{\theta}_{t,s}(\xi) = F(\theta_{t,s}(\xi)) + \Gamma \xi_1, \quad \theta_{t,t}(\xi) = \xi.$

Regularization by stochastic drift : proof. Derivatives estimates

$$
\begin{aligned}
u_i(t,x_1,x_2) \quad &= \quad \int_t^T \Big[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} F\Big](s,x_1,x_2)\mathbf{1}_{i=2} \\
&\quad + \Big[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} (\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(\xi)}) u_i\Big](s,x_1,x_2)\mathrm{d}s
\end{aligned}
$$

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Regularization by stochastic drift : proof. Derivatives estimates

$$D^{?}_{?}u_2(t,x_1,x_2) = D^{?}_{?}\int_t^T \Big[\tilde P^{\theta_{t,s}(\xi)}_{t,s}F\Big](s,x_1,x_2) + \Big[\tilde P^{\theta_{t,s}(\xi)}_{t,s}(\mathcal{L}-\tilde{\mathcal{L}}^{\theta_{t,s}(\xi)})u_2\Big](s,x_1,x_2)\mathrm{d}s$$

- Estimation of $(D_{x_1}+D_{x_2}+D^2_{x_1}+D_{x_1}D_{x_2})u_2$
- Fix $\xi=(\xi_1,\xi_2)$ $\to \theta$ solves $\dot\theta_{t,s}(\xi)=F(\theta_{t,s}(\xi))+\Gamma\xi_1, \quad \theta_{t,t}(\xi)=\xi.$

Regularization by stochastic drift : proof. Derivatives estimates

$$\begin{aligned} D_?^? u_2(t,x_1,x_2) &= \int_t^T D_?^? \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} F\right](s,x_1,x_2) \\ &\quad + D_?^? \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} (\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(\xi)}) u_2\right](s,x_1,x_2) \mathrm{d}s \end{aligned}$$

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- Degenerate Gaussian bound :

$$\forall \varphi \in \mathcal{B}(\mathbb{R}), \ \left| D_{x_1}^n D_{x_2}^m \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} \varphi\right](s,x_1,x_2) \right| \leqslant (s-t)^{-n/2-3m/2} \left[\hat{P}_{t,s}^{\theta_{t,s}(\xi)} |\varphi|\right](s,x_1,x_2)$$

Regularization by stochastic drift : proof. Derivatives estimates

$$D_?^? u_2(t,x_1,x_2) = \int_t^T D_?^? \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} F\right](s,x_1,x_2) + D_?^? \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} (\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(\xi)}) u_2\right](s,x_1,x_2) \mathrm{d}s$$

- Estimation of $(D_{x_1} + D_{x_2} + D_{x_1}^2 + D_{x_1} D_{x_2}) u_2$
- "Worst" case : $D_{x_1} D_{x_2} u_2$
- Fix $\xi = (\xi_1, \xi_2) \rightarrow \theta$ solves $\dot{\theta}_{t,s}(\xi) = F(\theta_{t,s}(\xi)) + \Gamma \xi_1, \quad \theta_{t,t}(\xi) = \xi.$

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Regularization by stochastic drift : proof. Derivatives estimates

$$D_{x_1} D_{x_2} u_2(t, x_1, x_2) = \int_t^T D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} F\right](s, x_1, x_2) + D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} (\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(\xi)}) u_2\right](s, x_1, x_2) \mathrm{d}s$$

- "Worst" case : $D_{x_1} D_{x_2} u_2$
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$$\forall \varphi \in \mathcal{B}(\mathbb{R}), \ \left| D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} \varphi\right] (s, x_1, x_2)\right| \leqslant (s - t)^{-2} \left[\hat{P}_{t,s}^{\theta_{t,s}(\xi)} |\varphi|\right] (s, x_1, x_2)$$

↳ the time singularity is not integrable !

Regularization by stochastic drift : proof. Derivatives estimates

$$D_{x_1} D_{x_2} u_2(t, x_1, x_2) = \int_t^T D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} F\right](s, x_1, x_2) + D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} (\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(\xi)}) u_2\right](s, x_1, x_2) \mathrm{d}s$$

- "Worst" case : $D_{x_1} D_{x_2} u_2$
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$$\forall \varphi \in \mathcal{B}(\mathbb{R}), \ \left| D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} \varphi\right](s, x_1, x_2)\right| \leqslant (s-t)^{-2} \left[\hat{P}_{t,s}^{\theta_{t,s}(\xi)} |\varphi|\right](s, x_1, x_2)$$

↳ the time singularity is not integrable !

- Smooth the singularity thanks to regularity of coefficients + Gaussian smoothing :
  ↳ $|F(\cdot) - F(\theta_{t,s}(\xi))| \leqslant |\cdot - \theta_{t,s}(\xi)|^\beta; \qquad |(\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(\xi)}) u_2| \leqslant |\cdot - \theta_{t,s}(\xi)|^\beta ||D_{x_2} u_2||_\infty$
  ↳ $\forall \gamma > 0, \ \left| \left[\hat{P}_{t,s}^{\theta_{t,s}(x)} |\cdot - \theta_{t,s}(x)|^\gamma\right](s, x_1, x_2)\right| \leqslant C(s-t)^{3\gamma/2}$

Regularization by stochastic drift : proof. Derivatives estimates

$$D_{x_1} D_{x_2} u_2(t, x_1, x_2) = \int_t^T D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} F \right](s, x_1, x_2) + D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} (\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(\xi)}) u_2 \right](s, x_1, x_2) \mathrm{d}s$$

- "Worst" case : $D_{x_1} D_{x_2} u_2$
- Fix $\xi = (\xi_1, \xi_2) \rightarrow \theta$ solves $\dot{\theta}_{t,s}(\xi) = F(\theta_{t,s}(\xi)) + \Gamma\xi_1, \quad \theta_{t,t}(\xi) = \xi.$
- ?
- Degenerate Gaussian bound :

$$\forall \varphi \in \mathcal{B}(\mathbb{R}), \ \left| D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} \varphi \right] (s, x_1, x_2) \right| \leqslant (s-t)^{-2} \left[\hat{P}_{t,s}^{\theta_{t,s}(\xi)} |\varphi| \right] (s, x_1, x_2)$$

↳ the time singularity is not integrable !

- Smooth the singularity thanks to regularity of coefficients + Gaussian smoothing :

↳ $|F(\cdot) - F(\theta_{t,s}(\xi))| \leqslant |\cdot - \theta_{t,s}(\xi)|^\beta; \qquad |(\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(\xi)}) u_2| \leqslant |\cdot - \theta_{t,s}(\xi)|^\beta ||D_{x_2} u_2||_\infty$

↳ $\forall \gamma > 0, \ \left| \left[\hat{P}_{t,s}^{\theta_{t,s}(x)} |\cdot - \theta_{t,s}(x)|^\gamma \right] (s, x_1, x_2) \right| \leqslant C(s-t)^{3\gamma/2}$

Regularization by stochastic drift : proof. Derivatives estimates

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- "Worst" case : $D_{x_1}D_{x_2}u_2$
- Fix $\xi=(\xi_1,\xi_2)$ $\rightarrow$ θ solves $\dot\theta_{t,s}(\xi)=F(\theta_{t,s}(\xi))+\Gamma\xi_1, \quad \theta_{t,t}(\xi)=\xi.$
- Use centering argument : $\forall\varphi\in\mathcal{B}(\mathbb{R}),\forall\zeta\in\mathbb{R},\ D_{x_1}D_{x_2}\left[\tilde P_{t,s}^{\theta_{t,s}(\xi)}\varphi\right](s,x_1,x_2) = D_{x_1}D_{x_2}\left[\hat P_{t,s}^{\theta_{t,s}(\xi)}(\varphi(\cdot)-\varphi(\zeta))\right](s,x_1,x_2)$
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Regularization by stochastic drift : proof. Derivatives estimates

$$D_{x_1} D_{x_2} u_2(t, x_1, x_2) = \int_t^T D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} (F(\cdot) - F(\theta_{t,s}(\xi)))\right](s, x_1, x_2) + D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} (\mathcal{L} - \tilde{\mathcal{L}}^{\theta_{t,s}(\xi)}) u_2\right](s, x_1, x_2) \mathrm{d}s$$

- "Worst" case : $D_{x_1} D_{x_2} u_2$
- Fix $\xi = (\xi_1, \xi_2) \rightarrow \theta$ solves $\dot{\theta}_{t,s}(\xi) = F(\theta_{t,s}(\xi)) + \Gamma \xi_1, \quad \theta_{t,t}(\xi) = \xi.$
- Use centering argument : $\forall \varphi \in \mathcal{B}(\mathbb{R}), \forall \zeta \in \mathbb{R},\ D_{x_1} D_{x_2} \left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)} \varphi\right](s, x_1, x_2) =$ $D_{x_1} D_{x_2} \left[\hat{P}_{t,s}^{\theta_{t,s}(\xi)} (\varphi(\cdot) - \varphi(\zeta))\right](s, x_1, x_2)$
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  ↳ $\forall \gamma > 0,\ \left|\left[\hat{P}_{t,s}^{\theta_{t,s}(x)} |\cdot - \theta_{t,s}(x)|^\gamma\right](s, x_1, x_2)\right| \leqslant C(s-t)^{3\gamma/2}$

Regularization by stochastic drift : proof. Derivatives estimates

$$\begin{aligned}\left|D_{x_1}D_{x_2}u_2(t,x_1,x_2)\right| \quad \leqslant \quad & \int_t^T \left|D_{x_1}D_{x_2}\left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)}(F(\cdot)-F(\theta_{t,s}(\xi)))\right](s,x_1,x_2)\right| \\ & +\left|D_{x_1}D_{x_2}\left[\tilde{P}_{t,s}^{\theta_{t,s}(\xi)}(\mathcal{L}-\tilde{\mathcal{L}}^{\theta_{t,s}(\xi)})u_2\right](s,x_1,x_2)\right|\mathrm{d}s\end{aligned}$$

- "Worst" case : $D_{x_1}D_{x_2}u_2$
- Fix $\xi=(\xi_1,\xi_2)$ $\rightarrow$ θ solves $\dot{\theta}_{t,s}(\xi)=F(\theta_{t,s}(\xi))+\Gamma\xi_1, \quad \theta_{t,t}(\xi)=\xi.$
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  ↳ $\forall\gamma>0,\ \left|\left[\hat{P}_{t,s}^{\theta_{t,s}(x)}|\cdot-\theta_{t,s}(x)|^{\gamma}\right](s,x_1,x_2)\right|\leqslant C(s-t)^{3\gamma/2}$

Regularization by stochastic drift : proof. Derivatives estimates

$$\left|D_{x_1} D_{x_2} u_2(t,x_1,x_2)\right| \leqslant \int_t^T (s-t)^{-2}\left[\hat{P}_{t,s}^{\theta_{t,s}(\xi)}|F(\cdot)-F(\theta_{t,s}(\xi))|\right](s,x_1,x_2)$$
$$+(s-t)^{-2}\left[\hat{P}_{t,s}^{\theta_{t,s}(\xi)}|(\mathcal{L}-\tilde{\mathcal{L}}^{\theta_{t,s}(\xi)})u_2|\right](s,x_1,x_2)\mathrm{d}s$$

- "Worst" case : $D_{x_1} D_{x_2} u_2$
- Fix $\xi=(\xi_1,\xi_2) \rightarrow \theta$ solves $\dot{\theta}_{t,s}(\xi)=F(\theta_{t,s}(\xi))+\Gamma\xi_1, \quad \theta_{t,t}(\xi)=\xi.$
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Regularization by stochastic drift : proof. Derivatives estimates

$$\Big|D_{x_1}D_{x_2}u_2(t,x_1,x_2)\Big| \leqslant \int_t^T (s-t)^{-2}\Big[\hat P_{t,s}^{\theta_{t,s}(\xi)}|F(\cdot)-F(\theta_{t,s}(\xi))|\Big](s,x_1,x_2)$$
$$+(s-t)^{-2}\Big[\hat P_{t,s}^{\theta_{t,s}(\xi)}|(\mathcal{L}-\tilde{\mathcal{L}}^{\theta_{t,s}(\xi)})u_2|\Big](s,x_1,x_2)\mathrm{d}s$$

- "Worst" case : $D_{x_1}D_{x_2}u_2$
- Fix $\xi=(\xi_1,\xi_2) \ \rightarrow \theta$ solves $\dot\theta_{t,s}(\xi)=F(\theta_{t,s}(\xi))+\Gamma\xi_1, \quad \theta_{t,t}(\xi)=\xi$.
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Regularization by stochastic drift : proof. Derivatives estimates

$$\Big|D_{x_1}D_{x_2}u_2(t,x_1,x_2)\Big| \leqslant \int_t^T (s-t)^{-2}\left[\hat{P}_{t,s}^{\theta_{t,s}(\xi)}|\cdot-\theta_{t,s}(\xi)|^\beta\right](s,x_1,x_2)$$

$$+||D_{x_2}u_2||_\infty(s-t)^{-2}\left[\hat{P}_{t,s}^{\theta_{t,s}(\xi)}|\cdot-\theta_{t,s}(\xi)|^\beta\right](s,x_1,x_2)\mathrm{d}s$$

- "Worst" case : $D_{x_1}D_{x_2}u_2$
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Regularization by stochastic drift : proof. Derivatives estimates

$$\left|D_{x_1} D_{x_2} u_2(t, x_1, x_2)\right| \leqslant \int_t^T (s-t)^{-2} \left[\hat{P}_{t,s}^{\theta_{t,s}(x)} |\cdot - \theta_{t,s}(x)|^\beta\right](s, x_1, x_2)$$

$$+||D_{x_2} u_2||_\infty (s-t)^{-2} \left[\hat{P}_{t,s}^{\theta_{t,s}(x)} |\cdot - \theta_{t,s}(x)|^\beta\right](s, x_1, x_2) \mathrm{d}s$$

- "Worst" case : $D_{x_1} D_{x_2} u_2$
- Choose $\xi = x$ $\quad \to \theta$ solves $\dot{\theta}_{t,s}(x) = F(\theta_{t,s}(x)) + \Gamma x_1, \quad \theta_{t,t}(x) = x_2$.
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Regularization by stochastic drift : proof. Derivatives estimates

$$\left|D_{x_1}D_{x_2}u_2(t,x_1,x_2)\right| \leqslant \int_t^T (s-t)^{-2+3\beta/2} + ||D_{x_2}u_2||_\infty (s-t)^{-2+3\beta/2}\mathrm{d}s$$

- "Worst" case : $D_{x_1}D_{x_2}u_2$
- Choose $\xi = x \quad \to \theta$ solves $\dot{\theta}_{t,s}(x) = F(\theta_{t,s}(x)) + \Gamma x_1, \quad \theta_{t,t}(x) = x_2$.
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Regularization by stochastic drift : proof. Derivatives estimates

$$\left|D_{x_1} D_{x_2} u_2(t, x_1, x_2)\right| \leqslant \int_t^T (s-t)^{-2+3\beta/2} + ||D_{x_2} u_2||_\infty (s-t)^{-2+3\beta/2} \mathrm{d}s$$

- “Worst” case : $D_{x_1} D_{x_2} u_2$
- $-2 + 3\beta/2 > -1 \Leftrightarrow \beta > 2/3 \rightarrow$ time singularity is integrable !

Regularization by stochastic drift : proof. Derivatives estimates

- We have $||D_{x_1} D_{x_2} u_2||_\infty \leqslant C T^{-1+3\beta/2}(1 + ||D_{x_2} u_2||_\infty)$

Regularization by stochastic drift : proof. Derivatives estimates

- We have $||D_{x_1} D_{x_2} u_2||_\infty \leqslant C T^{-1+3\beta/2}(1 + ||D_{x_2} u_2||_\infty)$
- And with the same arguments $||D_{x_2} u_2||_\infty \leqslant C T^{(-1+3\beta)/2}(1 + ||D_{x_2} u_2||_\infty)$

Regularization by stochastic drift : proof. Derivatives estimates

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Regularization by stochastic drift : proof. Derivatives estimates

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Regularization by stochastic drift : proof. Derivatives estimates

- We have $||D_{x_1} D_{x_2} u_2||_\infty \leqslant C'(T)$
- And with the same arguments $||D_{x_2} u_2||_\infty \leqslant C(T)$
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Regularization by stochastic drift : proof. Derivatives estimates

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Regularization by stochastic drift : proof. Derivatives estimates

- We have $||D_{x_1} D_{x_2} u_2||_\infty \leqslant C'(T)$
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  - And $C(T), C'(T) \to 0$ when $T \to 0$
- With similar arguments $||D^2_{x_1} u_2||_\infty + ||D_{x_1} u_2||_\infty \leqslant C(T)$

□

Regularization by stochastic drift : Conclusion

Consider on $[0,T]$, $T>0$, the stochastic system :

$$\begin{aligned} X_0^1 &= x_1, \quad \mathrm{d}X_t^1 = \mathrm{d}B_t \\ X_0^2 &= x_2, \quad \mathrm{d}X_t^2 = \big(F(X_t^2) + \Gamma X_t^1\big)\,\mathrm{d}t, \end{aligned}$$

where F is $\beta > 2/3$-Hölder and $\Gamma \neq 0$ is well-posed in a strong sense, thanks to strong theory for the associated PDE.

Regularization by stochastic drift : "strong" result

Consider on $[0,T]$, $T>0$, the hypoelliptic stochastic system :

$$\begin{aligned} X_0^1 &= x_1, \quad \mathrm{d}X_t^1 = F_1(t,X_t^1,X_t^2)\mathrm{d}t + \sigma(t,X_t^1,X_t^2)\mathrm{d}B_t \\ X_0^2 &= x_2, \quad \mathrm{d}X_t^2 = F_2(t,X_t^1,X_t^2)\mathrm{d}t \end{aligned}$$

Theorem. If :

- σ is Lipschitz and uniformly elliptic,
- $\forall (t,x_2) \in [0,T]\times\mathbb{R}^d, x_1 \mapsto D_{x_1}F_2(t,x_1,x_2)$ is differentiable and $D_{x_1}F_2 \in \mathcal{E} \subset \mathrm{GL_d}(\mathbb{R})$, $\mathcal{E}$ closed convex,
- F_1, F_2 Hölder, Hölder exponent (second variable) $> 2/3$
- $D_{x_1}F_2$ is Hölder in space,

then, the system is well-posed (strong sense).

Regularization by stochastic drift : "strong" result

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Theorem. If :

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- $D_{x_1} F_2$ is Hölder in space,

then, the system is well-posed (strong sense).

Regularization by stochastic drift : "strong" result

Consider on $[0, T]$, $T > 0$, the hypoelliptic stochastic system :

$$\begin{array}{ll} X_0^1 = x_1, & \mathrm{d}X_t^1 = F_1(t, X_t^1, X_t^2)\mathrm{d}t + \sigma(t, X_t^1, X_t^2)\mathrm{d}B_t \\ X_0^2 = x_2, & \mathrm{d}X_t^2 = F_2(t, X_t^1, X_t^2)\mathrm{d}t \end{array}$$

Theorem. If :

- σ is Lipschitz and uniformly elliptic,
- $\forall (t, x_2) \in [0, T] \times \mathbb{R}^d, x_1 \mapsto D_{x_1} F_2(t, x_1, x_2)$ is differentiable and $D_{x_1} F_2 \in \mathcal{E} \subset \mathrm{GL_d}(\mathbb{R})$, $\mathcal{E}$ closed convex,
- F_1, F_2 Hölder, Hölder exponent (second variable) $> 2/3$
- $D_{x_1} F_2$ is Hölder in space,

then, the system is well-posed (strong sense).

KOLMOGOROV EQUATION ON SPACE OF PROBABILITY MEASURES

and

REGULARIZATION BY NOISE FOR (a class of) McKEAN-VLASOV PROCESSES

McKean-Vlasov system

Consider on $[0, T]$, $T > 0$, the system

$$X_0 \sim \mu, \quad \mathrm{d}X_t = F(X_t, [X_t])\mathrm{d}t + \Sigma \mathrm{d}B_t, \quad \mu \in \mathcal{P}_2(\mathbb{R}^d)$$

where $(B_t)_t \geqslant 0$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$

- "$[\cdot]$" means that we are interested with the law : $\forall t \in [0, T]$, $[X_t]$ is the law of the r.v. X_t

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Consider on $[0, T]$, $T > 0$, the system

$$X_0 \sim \mu, \quad \mathrm{d}X_t = F(X_t, [X_t])\mathrm{d}t + \Sigma \mathrm{d}B_t, \quad \mu \in \mathcal{P}_2(\mathbb{R}^d)$$

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$$X_0^1 \sim \mu, \qquad \mathrm{d}X_t^1 = F(X_t^1, \qquad\qquad)\mathrm{d}t + \mathrm{d}B_t^1$$

$$\vdots$$

$$X_0^M \sim \mu, \qquad \mathrm{d}X_t^M = F(X_t^M, \qquad\qquad)\mathrm{d}t + \mathrm{d}B_t^M$$

- M players

- "$[\cdot]$" means that we are interested with the law : $\forall t \in [0, T]$, $[X_t]$ is the law of the r.v. X_t

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- M players
- $\forall i$, $[X_0^i] = \mu$, $B^1, \ldots, B^M$ independent B.M.
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- M players
- $\forall i$, $[X_0^i] = \mu$, $B^1, \ldots, B^M$ independent B.M. + indistinguishable dynamics
- "$[\cdot]$" means that we are interested with the law : $\forall t \in [0,T]$, $[X_t]$ is the law of the r.v. X_t

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Consider on $[0, T]$, $T > 0$, the system

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where $(B_t)_t \geqslant 0$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$

$$\begin{aligned} X_0^1 &\sim \mu, \qquad \mathrm{d}X_t^1 = F(X_t^1, M^{-1}\sum_{i=1}^M \delta_{X_t^i})\mathrm{d}t + \mathrm{d}B_t^1 \\ &\vdots \\ X_0^M &\sim \mu, \qquad \mathrm{d}X_t^M = F(X_t^M, M^{-1}\sum_{i=1}^M \delta_{X_t^i})\mathrm{d}t + \mathrm{d}B_t^M \end{aligned}$$

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- $\forall i$, $[X_0^i] = \mu$, $B^1, \dots, B^M$ independent B.M. + indistinguishable dynamics
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$$\vdots$$

- M interacting players
- $\forall i,\ [X_0^i] = \mu,\ B^1, \ldots, B^M$ independent B.M. + indistinguishable dynamics
- $M \to +\infty$
- "$[\cdot]$" means that we are interested with the law : $\forall t \in [0, T]$, $[X_t]$ is the law of the r.v. X_t

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Consider on $[0,T]$, $T>0$, the system

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- M interacting players
- $\forall i$, $[X_0^i] = \mu$, $B^1, \ldots, B^M$ independent B.M. + indistinguishable dynamics
- $M \to +\infty \to$ Propagation of chaos + LLN
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- Condition for (strong) well posedness
  ↳ Sniztmann : Lipschitz in space and w.r.t. the Wasserstein distance for the law argument

$$d(\mu, \nu) = \inf_{Y \sim \mu,\ Y' \sim \nu} \mathbb{E}[|X - Y|].$$

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  ↳ Weak well posedness (Jourdain) : bounded drift in space and Lipschitz for total variation distance w.r.t. measure argument, $\Sigma =$ Id

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- Specify the law dependence

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Consider on $[0, T]$, $T > 0$, the system

$$X_0 \sim \mu, \quad \mathrm{d}X_t = b(X_t, \langle \varphi_1, [X_t] \rangle)\mathrm{d}t + \mathrm{d}B_t, \quad \mu \in \mathcal{P}_2(\mathbb{R}^d)$$

where $(B_t)_t \geqslant 0$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ $\varphi_1 : \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}$

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- Specify the law dependence, structural assumption : polynomial dependence

$$F(x, \nu) = b(X_t, \langle \varphi_1, \nu \rangle).$$

$\langle \varphi, \nu \rangle = \int \varphi \mathrm{d}\nu \to$ link with "linear" system :

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  ↳ Strong well posedness outside Cauchy Lipchitz framework ?

$$\forall x, y \in \mathbb{R},\ \forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}),\ |b(x,\mu) - b(y,\nu)| \leqslant C(|x-y|^{\alpha} + \inf_{X\sim\mu,\ Y\sim\nu} \mathbb{E}[|X-Y|^{\alpha_1}])$$

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↳ Regularization by noise phenomenon

Regularization by noise : McKean-Vlasov processes, result

Consider on $[0, T]$, $T > 0$, the system

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where $(B_t)_t \geqslant 0$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ $(\Omega, \mathcal{A}, \mathbb{P})$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, σ elliptic

- **Proposition :** If b is Hölder and bounded, differentiable w.r.t. its second argument with Hölder derivative, then, strong existence and uniqueness hold as soon as φ_1 is Hölder.

Consider on $[0, T]$, $T > 0$, the system

$$X_0 \sim \mu, \quad \mathrm{d}X_t = \langle b(X_t, \cdot), [X_t] \rangle \mathrm{d}t + \sigma \mathrm{d}B_t, \quad \mu \in \mathcal{P}_2(\mathbb{R})$$

where $(B_t)_t \geqslant 0$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ $(\Omega, \mathcal{A}, \mathbb{P})$, σ elliptic

- **Corollary :** If b is Hölder continuous and bounded then strong existence and uniqueness hold.

Regularization by noise : linear case

Consider on $[0, T]$, $T > 0$, the system

$$X_0 = x, \quad \mathrm{d}X_t = F(X_t)\mathrm{d}t + \Sigma \mathrm{d}B_t$$

where $(B_t)_t \geqslant 0$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ $(\Omega, \mathcal{A}, \mathbb{P})$, Σ elliptic.

- Well posed (strong sense) for F α-Hölder (F bounded : Zvonkin, Veretennikov, Krylov et Röckner).
- Why stochastic perturbation regularizes the system ?

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  ↳ Zvonkin transformation
  ↳ F writes :

$$\int_0^t F(X_s)\mathrm{d}s = u(t, X_t) - u(0, x) - \int_0^t \partial_x u(s, X_s)\mathrm{d}B_s$$

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Regularization by noise : linear case

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↳ Σ elliptic $\rightarrow$ the process Xvisits all the space around the initial condition $\rightarrow$ mixes the values of harmonic function associated to the operator $\partial_t + \mathcal{L}$ $\rightarrow$ and regularizes it

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Existence and uniqueness on small time intervals

Regularization by noise : linear case

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Existence and uniqueness on small time intervals $\overset{\text{"Markov"}}{\longrightarrow}$ $\exists! \ \mathbb{R}^+$

Regularization by noise : McKean-Vlasov processes, proof

Consider on $[0,T]$, $T>0$, the system

$$X_0 \sim \mu, \quad \mathrm{d}X_t = b(X_t, \langle \varphi_1, [X_t]\rangle)\mathrm{d}t + \sigma \mathrm{d}B_t, \quad \mu \in \mathcal{P}_2(\mathbb{R})$$

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↳ Non-Markovian

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$$Z \in \mathbb{L}_2, \; V([Z]) \to \mathcal{V}(Z),$$

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↳ $Z \in \mathbb{L}_2, \quad V([Z]) = \mathbb{E}[\varphi(Z)] \overset{Taylor}{\to} V([Z + \epsilon H]) = \mathbb{E}[\varphi(Z)] + \epsilon \mathbb{E}[D\varphi(Z) \cdot H] + o(\epsilon)$

↳ $\partial_{[Z]} \langle \varphi, [Z] \rangle = \varphi'$

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↳ $\forall \psi$ sufficiently smooth : $\mathcal{A}\psi(t, x, \mu) :=$

$$(\frac{1}{2}\Delta + b(x)\partial_x)\psi(t, x, \mu) + \int b(z)\partial_\mu \psi(t, x, \mu)(z)\mathrm{d}\mu(z) + \frac{1}{2}\int \partial_z \partial_\mu \psi(t, x, \mu)(z)\mathrm{d}\mu(z)$$

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  "linear" version of the "Master equation" (Buckdahn, Li, Peng / Chassagneux, Crisan, Delarue / Lasry, Lions)

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$$\begin{cases} X_t^{t,\mu} \sim \mu, & \mathrm{d}X_s^{t,\mu} = b(X_s^{t,\mu}, \langle \varphi_1, [X_s^{t,\mu}] \rangle)\mathrm{d}s + \sigma \mathrm{d}B_s \\ X_t^{t,x,\mu} = x, & \mathrm{d}X_s^{t,x,\mu} = b(X_s^{t,x,\mu}, \langle \varphi_1, [X_s^{t,\mu}] \rangle)\mathrm{d}s + \sigma \mathrm{d}B_s \end{cases}$$

where $(B_t)_t \geqslant 0$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, σ elliptic

- Smooth setting $\to$ $\exists!$ solution $\to$ family of marginals $([X_s^{t,\mu}])_{t \leqslant s \leqslant T}$
- Freeze the law in the flow $\to$ $X^{t,x,\mu}$ has a transition density $p(\mu; t, x; \cdot, \cdot) \approx$ Gaussian
- $u(t, x, \mu) = \mathbb{E} \int_t^T b(X_s^{t,x,\mu}, \langle \varphi_1, [X_s^{t,\mu}] \rangle)\mathrm{d}s$
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↳ spatial regularization

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$$\begin{cases} X_t^{t,\mu} \sim \mu, & \mathrm{d}X_s^{t,\mu} = b(X_s^{t,\mu}, \langle \varphi_1, [X_s^{t,\mu}] \rangle)\mathrm{d}s + \sigma \mathrm{d}B_s \\ X_t^{t,x,\mu} = x, & \mathrm{d}X_s^{t,x,\mu} = b(X_s^{t,x,\mu}, \langle \varphi_1, [X_s^{t,\mu}] \rangle)\mathrm{d}s + \sigma \mathrm{d}B_s \end{cases}$$

where $(B_t)_t \geqslant 0$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, σ elliptic

- Smooth setting $\to \exists!$ solution $\to$ family of marginals $([X_s^{t,\mu}])_{t \leqslant s \leqslant T}$
- Freeze the law in the flow $\to$ $X^{t,x,\mu}$ has a transition density $p(\mu; t, x; \cdot, \cdot) \approx$ Gaussian
- $u(t, x, \mu) = \mathbb{E} \int_t^T b(X_s^{t,x,\mu}, \langle \varphi_1, [X_s^{t,\mu}] \rangle)\mathrm{d}s = \int_t^T \int_{\mathbb{R}} b(y, \langle \varphi_1, [X_s^{t,\mu}] \rangle) p(\mu; t, x; s, y)\mathrm{d}y\mathrm{d}s$
- Estimation of $||\partial_\mu u||_\infty$, $||\partial_\mu \partial_x u||_\infty$ $\to \partial_\mu(\partial_x)p(\mu; t, x; s, y)$ et $\partial_\mu \langle \varphi_1, [X_s^{t,\mu}] \rangle$

  ↳ $p(\mu; t, x; s, y)$ depends on $\langle \varphi_1, [X_s^{t,\mu}] \rangle$ $\to$ Gaussian bound for $\partial_1 p$?
- Regularization on the initial data μ for $s > t$

$$\forall z \in \mathbb{R}, \ |(\partial_\mu \langle \varphi_1, [X_s^{t,\mu}] \rangle)(z)| \leqslant C(s - t)^{(\alpha_1 - 1)/2} ?$$

↳ spatial regularization

- Density estimates

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- Process $X_s^{t,x,\mu} \overset{\text{Frozen law}}{\Longrightarrow}$ Gaussian
- Gaussian transition density :

$$p(\mu; t, x; s, y) = \frac{1}{\sigma(2\pi(s-t))^{1/2}} \exp\left(-\frac{1}{2\sigma(s-t)^{1/2}} \left(y - x - \int_t^s b(\langle \varphi_1, [X_r^{t,\mu}] \rangle)\mathrm{d}r\right)^2 \right)$$

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- Gaussian estimate on the law derivative : for all z in $\mathbb{R}$

$$\partial_\mu p(\mu; t, x; s, y)(z)$$

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$$\begin{aligned} &\partial_\mu p(\mu; t, x; s, y)(z) \\ &= \frac{1}{(s-t)^{1/2}} \left(\partial_\mu \int_t^s b(\langle \varphi_1, [X_r^{t,\mu}]\rangle)\mathrm{d}r\right)(z) \frac{1}{\sigma(s-t)^{1/2}}\left(y - x - \int_t^s b(\langle \varphi_1, [X_r^{t,\mu}]\rangle)\mathrm{d}r\right) \\ &\quad \times \frac{1}{\sigma(2\pi(s-t))^{1/2}} \exp\left(-\frac{1}{2\sigma(s-t)^{1/2}}\left(y - x - \int_t^s b(\langle \varphi_1, [X_r^{t,\mu}]\rangle)\mathrm{d}r\right)^2\right) \end{aligned}$$

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$$\begin{aligned}
&\left|\partial_\mu p(\mu; t, x; s, y)(z)\right| \\
&= \Bigg|\frac{1}{(s-t)^{1/2}}\left(\partial_\mu \int_t^s b(\langle \varphi_1, [X_r^{t,\mu}]\rangle)\mathrm{d}r\right)(z)\frac{1}{\sigma(s-t)^{1/2}}\left(y - x - \int_t^s b(\langle \varphi_1, [X_r^{t,\mu}]\rangle)\mathrm{d}r\right) \\
&\quad \times \frac{1}{\sigma(2\pi(s-t))^{1/2}} \exp\left(-\frac{1}{2\sigma(s-t)^{1/2}}\left(y - x - \int_t^s b(\langle \varphi_1, [X_r^{t,\mu}]\rangle)\mathrm{d}r\right)^2\right)\Bigg| \\
&\leqslant C||b'||_\infty (s-t)^{\alpha_1/2} \sup_{r\in[t,s]} |(r-t)^{(1-\alpha_1)/2}\partial_\mu\langle \varphi_1, [X_r^{t,\mu}]\rangle(z)| g(s-t, y-x)
\end{aligned}$$

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  ↳ $\partial_\mu(\partial_x)p(\mu; t, x; s, y) \to$ Gaussian bounds

  ↳ $\partial_\mu \langle \varphi_1, [X_s^{t,\mu}] \rangle$

- $\partial_x \to$ time singularity of order $1/2$ | $\partial_\mu \to$ time singularity of order $1/2 - \alpha_1/2$

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- $\partial_x \to$ time singularity of order $1/2$ | $\partial_\mu \to$ time singularity of order $1/2 - \alpha_1/2$
- There exist $C(T), C'(T)$, independent of the regularization and such that $C(T), C'(T) \to 0$ when $T \to 0$ such that

$$|\partial_\mu u(t, x, \mu)(z)| \leqslant C(T)$$
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- Zvonkin transformation : the system is well posed (strong sense)

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where $(B_t)_{t\geqslant 0}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\varphi_1, \varphi_2 : \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}$, $\sigma\sigma^*$ uniformly elliptic

Theorem : If b is Hölder, bounded and smooth (Lipschitz) w.r.t. its third argument, if φ_1 is α_1-Hölder, if σ is Hölder in space, smooth w.r.t. its third argument φ_2 Hölder, then : the regularized solution of the linear equation

$$\partial_t u(t,x,\mu) + \mathcal{A}u(t,x,\mu) = b(x, \langle \varphi_1, \mu \rangle) \text{ sur } [0,T) \times \mathbb{R} \times \mathcal{P}(\mathbb{R}^d), \quad u(T,x,\mu) = 0$$

where $\forall \psi$ sufficiently smooth :

$$\begin{aligned} \mathcal{A}\psi(t,x,\mu) &= \frac{1}{2}\mathrm{Tr}\left[a(t,x,\langle \mu, \varphi_2 \rangle)\partial_x^2 \psi(t,x,\mu)\right] + b(t,x,\langle \varphi_1, \mu \rangle)\partial_x \psi(t,x,\mu) \\ &+ \int b(t,x,\langle \varphi_1, \mu \rangle)\partial_\mu \psi(t,x,\mu)(z)\mathrm{d}\mu(z) \\ &+ \frac{1}{2}\int \mathrm{Tr}\left[a(t,x,\langle \varphi_2, \mu \rangle)\partial_z(\partial_\mu \psi(t,x,\mu)(z))\right]\mathrm{d}\mu(z). \end{aligned}$$

satisfies

$$|\partial_\mu u(t,x,\mu)(z)| + |\partial_\mu(\partial_x u(t,x,\mu)(z))| + |\partial_x u(t,x,\mu)| + \left|\partial_x^2 u(t,x,\mu)\right| \leqslant CT^\delta,$$

where C, $\delta > 0$ do not depend on the regularization procedure.

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- **Theorem :** If b Hölder, bounded and smooth (Lipschitz) in its third argument, if φ_1 est α_1-Hölder, then the McKean-Vlasov system is well posed

- **Corollary :** If b is Hölder, bounded and smooth in its third argument, if φ_1 is α_1-Hölder, then for all function ϕ Hölder , for all $s > t$ the function

  $$\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \langle \phi, [X_s^{t,\mu}] \rangle$$

  is differentiable (Lions).

Regularization by noise : McKean-Vlasov processes, result

Consider on $[0, T]$ the system

$$\begin{cases} X_t^{t,\mu} \sim \mu, & \mathrm{d}X_s^{t,\mu} = \langle b(s, X_s^{t,\mu}, \cdot), [X_s^{t,\mu}]\rangle \mathrm{d}s + \langle \sigma(X_s^{t,\mu}, \cdot), [X_s^{t,\mu}]\rangle \mathrm{d}B_s \\ X_t^{t,x,\mu} = x, & \mathrm{d}X_s^{0,x,\mu} = \langle b(s, X_s^{t,x,\mu}, \cdot), [X_s^{t,\mu}]\rangle \mathrm{d}s + \langle \sigma(X_s^{t,\mu}, \cdot), [X_s^{t,\mu}]\rangle \mathrm{d}B_s \end{cases}$$

where $(B_t)_{t\geqslant 0}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\sigma\sigma^*$ uniformly elliptic

- **Corollary :** If b is bounded and Hölder and σ Lipschitz in its arguments, then the McKean-Vlasov system is well posed

Thank you !