Sharp Estimates for Fundamental Solutions of some degenerate Kolmogorov equations arising in Finance.

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Aim of the work

The aim of the work is to prove optimal lower and upper bounds for the Fundamental solution $\Gamma(x, y, t)$ of the following parabolic degenerate operator

$$\mathscr{L}u = x\partial_x(a(x, y, t)x\partial_x u) + b(x, y, t)x\partial_x u + x\partial_y u - \partial_t u,$$

where $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times (0, T)$.

The functions $a(x, y, t), b(x, y, t)$ belong to $L^\infty(\mathbb{R}^+ \times \mathbb{R} \times (0, T))$ and satisfy suitable regularity conditions and $a(x, y, t)$ is bounded by below.

Note that we can rewrite

$$\mathscr{L}u = ax^2\partial_x^2 u + (b + a + a_x x)x\partial_x u + x\partial_y u - \partial_t u$$

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Examples

Uniformly Parabolic Operators: $\mathcal{L}_0 = \partial_{xx} + \partial_{yy} - \partial_t = X_1^2 + X_2^2 - Y$, where the vector fields

$$X_1 = \partial_x \sim (1,0,0), \; X_2 = \partial_y \sim (0,1,0), \; Y = -\partial_t \sim (0,0,-1)$$

form a basis of $\mathbb{R}^3$ for every $(x,y,t) \in \mathbb{R}^2 \times (0,T)$.

Hypoelliptic Operators: Operators whose vector fields do not form a basis of $\mathbb{R}^3$ for every $(x,y,t) \in \mathbb{R}^2 \times (0,T)$:

Kolmogorov Operators: $\mathcal{K}_0 = \partial_{xx} + x\partial_y - \partial_t = X_1^2 + X_0 - \partial_t$,
where $X_1 = \partial_x \sim (1,0,0)$, $Y = X_0 - \partial_t = x\partial_y - \partial_t \sim (0,x,-1)$,

Asian Option Operators: $\mathscr{L}_0 = x^2\partial_{xx} + x\partial_x + x\partial_y - \partial_t = X_1^2 + X_0 - \partial_t$,
where $X_1 = x\partial_x \sim (x,0,0)$, $Y = X_0 - \partial_t = x\partial_y - \partial_t \sim (0,x,-1)$.

Hypoellipticity and Hörmander Theorem

Definition (Hypoellipticity)

We say that an N-dimensional operator

$$\mathcal{L} = \sum_{i=1}^{m} X_i^2 + X_0 - \partial_t$$

is hypoelliptic if for every distributional solution u of $\mathcal{L}u = f$ in $\Omega \subseteq \mathbb{R}^N$, we have

$$u \in C^\infty(\Omega) \quad \text{whenever} \quad f \in C^\infty(\Omega).$$

Theorem (Hörmander 1967)

If $\mathrm{Lie}\{X_1, \ldots, X_m, Y\}(x,t) = \mathbb{R}^{N+1}$ *at every* $(x,t) \in \Omega$*, then* $\mathcal{L}$ *is hypoelliptic.*

Hypoellipticity of $\mathcal{K}_0$ and $\mathcal{L}_0$

Kolmogorov Operators: $\mathcal{K}_0 = \partial_{xx} + x\partial_y - \partial_t = X_1^2 + Y$ and

$$X_1 = \partial_x \sim (1,0,0), \quad Y = x\partial_y - \partial_t \sim (0,x,-1),$$
$$[X_1, Y] = \partial_x(x\partial_y - \partial_t) - (x\partial_y - \partial_t)\partial_x = \partial_y \sim (0,1,0).$$

Hence, the vector fields $(1,0,0)$, $(0,x,-1)$, $(0,1,0)$ form a basis of $\mathbb{R}^3$ for every $(x,y,t) \in \mathbb{R}^2 \times (0,T)$.

Asian Option Operators: $\mathcal{L}_0 = x^2\partial_{xx} + x\partial_x + x\partial_y - \partial_t = X_1^2 + Y$,and

$$X_1 = x\partial_x \sim (x,0,0), \quad Y = x\partial_y - \partial_t \sim (0,x,-1),$$
$$[X_1, Y] = x\partial_x(x\partial_y - \partial_t) - (x\partial_y - \partial_t)x\partial_x = x\partial_y \sim (0,x,0).$$

Hence the vector fields $(x,0,0)$, $(0,x,-1)$, $(0,x,0)$ form a basis of $\mathbb{R}^3$ for every $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times (0,T)$.

Asian Option Operator Vs variable coefficients Kolmogorov

A Kolmogorov operator with variable coefficients is:

$$\mathcal{K} = \widetilde{a}(x, y, t)\partial_{xx} + \widetilde{b}(x, y, t)\partial_x + x\partial_y - \partial_t,$$

where $\widetilde{a}(x, y, t), \widetilde{b}(x, y, t)$ is a smooth bounded function and $\widetilde{a}(x, y, t)$ is bounded by below (with positive infimum).

Concerning the Asian Option operator

$$\mathscr{L}_0 = x^2\partial_{xx} + x\partial_x + x\partial_y - \partial_t, \quad (x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times (0, T),$$

the new difficulty is in managing the unbounded coefficients x^2 and x and the fact that the infimum of x^2 equals zero in $\mathbb{R}^+$.

Main result of the work

Theorem (C., Polidoro, Rossi)

Let Γ *be the fundamental solution of* $\mathscr{L}$. *Then for every* $(x_0, y_0, t_0), (x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]$ *we have*

$$\Gamma(x, y, t, x_0, y_0, t_0) = 0 \qquad \forall\, (x, y, t) \in \mathbb{R}^+ \times \mathbb{R}^2 \setminus \{]-\infty, y_0[\times]t_0, T[\}.$$

Moreover, for arbitrary $\varepsilon \in]0, 1/(4T)[$, *there exist four positive constants* $c_\varepsilon^-, C_\varepsilon^+, C^-, c^+$ *such that*

$$\frac{c_\varepsilon^-}{x_0^2(t-t_0)^2} \exp\left(-C^-\Psi(x, y + x_0\varepsilon(t-t_0), t - \varepsilon(t-t_0); x_0, y_0, t_0)\right) \leq$$

$$\Gamma(x, y, t; x_0, y_0, t_0) \leq$$

$$\frac{C_\varepsilon^+}{x_0^2(t-t_0)^2} \exp\left(-c^+\Psi(x, y - x_0\varepsilon, t + \varepsilon; x_0, y_0, t_0)\right),$$

for every $(x, y, t) \in \mathbb{R}^+ \times]-\infty, y_0 - x_0\varepsilon(t-t_0)[\times]t_0, T]$. *Here* Ψ *is the value function of a suitable Control Problem.*

Useful Application

The fundamental solution Γ_0 of $\mathscr{L}_0$ is known and has been first written by Yor (1992).

By applying the Theorem to Γ and to the fundamental solutions $\Gamma^{\pm}$ of the operators

$$\mathscr{L}^{\pm} u = \lambda^{\pm} x^2 \partial_{xx} u + x \partial_x u + x \partial_y u - \partial_t u, \quad (x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times (0, T),$$

we obtain (set $\Gamma(x, y, t) = \Gamma(x, y, t; 1, 0, 0)$)

$$\begin{aligned} k_\varepsilon^- \Gamma^- \big(x, y + \varepsilon(t+1), t - \varepsilon(t+1)\big) &\leq \Gamma(x, y, t) \\ &\leq k_\varepsilon^+ \Gamma^+ \left(x, y - \tfrac{\varepsilon}{1-\varepsilon}(t+1), t + \tfrac{\varepsilon}{1-\varepsilon}(t+1)\right), \end{aligned}$$

where $y + \varepsilon(t+1) < 0$ and $t > \varepsilon/(1-\varepsilon)$.

Available results in Parabolic PDE's Theory

Let $(x,t),(\xi,\tau) \in \mathbb{R}^N \times (0,T)$ with $\tau < t$:

1. **Uniformly parabolic operator**: $L = \sum_{i,j=1}^N \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}\right) - \partial_t$:

$$\frac{c^-}{(t-\tau)^{N/2}} \exp\left(-C^- \frac{|x-\xi|^2}{t-\tau}\right) \le \Gamma(x,t;\xi,\tau) \le \frac{C^+}{(t-\tau)^{N/2}} \exp\left(-c^+ \frac{|x-\xi|^2}{t-\tau}\right),$$

2. **Heat operator on Carnot groups**: Let $X_i(x) : x \in \mathbb{R}^N \mapsto \mathbb{R}$ be smooth vector fields for every $i = 1, ..., m$ and consider the operator $\mathcal{L} = \sum_{i=1}^m X_i^2(x) - \partial_t$:

$$\frac{1}{C\sqrt{|\mathcal{B}_{t-\tau}(x)|}} \exp\left(-\frac{C\, d_{CC}^2(x,\xi)}{t-\tau}\right) \le \Gamma(x,t;\xi,\tau) \le \frac{C}{\sqrt{|\mathcal{B}_{t-\tau}(x)|}} \exp\left(-\frac{d_{CC}^2(x,\xi)}{C\,(t-\tau)}\right),$$

where d_{CC} denotes the *Carnot-Caratheodory distance*;

3. **Kolmogorov operator**: $\mathcal{K} = \mathrm{div}\,(\langle A(x,t);\nabla u\rangle) + \sum_{i,j=1}^N b_{ij}x_i\partial_{x_j} - \partial_t$:

$$k^-\Gamma_0^-(x, t-\varepsilon(t-\tau);\xi,\tau) \le \Gamma(x,t;\xi,\tau) \le k^+\Gamma_0^+(x,t+\varepsilon(t-\tau);\xi,\tau).$$

Involved Techniques

Lower Bound key Ingredients: Harnack Inequality, Harnack Chains, Pontryagin Maximum Principle.

Upper Bound key Ingredients: Moser type estimates, Nash type estimates, H-J-B equation.

Existence of a Strong Solution and Uniqueness

Let $t \in [0, T]$ and consider the operator

$$\mathcal{L} = a(x,y,t)x^2\partial_{xx} + \big(a_x(x,y,t)x + a(x,y,t) + b(x,y,t)\big)x\partial_x + x\partial_y,$$

and assume that the functions a, b and are bounded and smooth. Assume also that $x\partial_x a$ is bounded.
By using the theory of Stochastic Processes we have that $\mathcal{L} + \partial_t$ is the infinitesimal generator of the process

$$\begin{cases} dX_t = \mu(X_t, Y_t, t)X_t dt + \sigma(X_t, Y_t, t)X_t dW_t \\ dY_t = X_t dt. \end{cases} \tag{1}$$

where $a = \frac{\sigma^2}{2}, \quad b + \frac{\sigma^2}{2} + \sigma\sigma_x x = \mu.$
Moreover, there exists a unique strong solution of the SDE (1).

Malliavin Calculus and existence of a smooth density

In this setting, provided that the coefficients σ, μ are smooth and bounded with bounded derivatives of any order, σ^2 has positive infimum, **Malliavin Calculus** guarantees the existence of a **smooth** fundamental solution of the operator $\mathcal{L} + \partial_t$ if it satisfies the **Hörmander condition**:

$$\begin{aligned} X &= x\sqrt{a}\partial_x & &\sim \quad (x\sqrt{a}, 0, 0); \\ Y &= (a_x x + a + b)x\partial_x + x\partial_y + \partial_t & &\sim \quad ((a_x x + a + b)x, x, 1); \\ [X, Y] &= x^2(\cdots)\partial_x + x\sqrt{a}\partial_y & &\sim \quad (x^2(\cdots), x\sqrt{a}, 0); \end{aligned}$$

hence, X, Y, $[X, Y]$ form a basis of $\mathbb{R}^3$ for every $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times (0, T)$.
Moreover, such fundamental solution belongs to the Schwartz space.

Lie Group structure for the model Operator

Let us consider the model operator

$$\mathscr{L}_0 = x^2\partial_{xx} + x\partial_x + x\partial_y - \partial_t$$

Observe that if we set

$$v(x, y, t) = u(x_0 x, y_0 + x_0 y, t_0 + t),$$

then $\mathscr{L}_0 v = 0$ if, and only if $\mathscr{L}_0 u = 0$. The algebraic structure $\mathbb{G} := (\mathbb{R}^+ \times \mathbb{R}^2, \circ)$ where

$$(x_0, y_0, t_0) \circ (x, y, t) = (x_0 x, y_0 + x_0 y, t_0 + t) \tag{2}$$

is a Lie group, its identity $\mathbf{1}_{\mathbb{G}}$ and the inverse of (x, y, t) are defined as

$$\mathbf{1}_{\mathbb{G}} = (1, 0, 0), \qquad (x, y, t)^{-1} = \left(\frac{1}{x}, -\frac{y}{x}, -t\right).$$

Our Assumptions on the coefficients

Consider the operator

$$\mathscr{L} u = x\partial_x(a(x,y,t)x\partial_x u) + b(x,y,t)x\partial_x u + x\partial_y u - \partial_t u,$$

1. We assume that $a, b, \partial_x(xa)$ and $\partial_x(xb)$ are bounded,
2. There exists $\lambda > 0$ such that
$$a(x,y,t) \geq \lambda \quad \text{for every} \quad (x,y,t),(\xi,\eta,\tau) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (0,T).$$
3. (Hölder continuity) there exist $\overline{M} \geq 0, \lambda \geq 1$ and $\alpha \in]0,1]$ such that
$$|a(x,y,t) - a(\xi,\eta,\tau)| \leq \overline{M} \left(\left|\frac{x-\xi}{\xi}\right| + \left|\frac{y-\eta}{\xi} + t - \tau\right|^{1/3} + |t-\tau|^{1/2} \right)^{\alpha},$$
for every $(x,y,t),(\xi,\eta,\tau) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (0,T)$. The same condition is assumed on $b, \partial_x(xa)$ and $\partial_x(xb)$.

Harnack inequality

Theorem (C.,Polidoro,Rossi)

Let $z_0 \in \mathbb{R}^+ \times \mathbb{R}^2$ *and* $r \in]0,1/2]$ *and consider the sets*

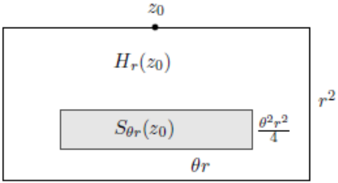
$$H_r(z_0) = \left\{(x,y,t) \in \mathbb{R}^3 : |x-x_0| < r x_0, -r^2 < t - t_0 < 0, |y - y_0 + x_0(t-t_0)| < r^3 x_0\right\}$$

$$S_r(z_0) = \left\{(x,y,t) \in \mathbb{R}^3 : |x-x_0| \le r x_0, -\frac{3r^2}{4} \le t - t_0 \le -\frac{r^2}{2}, |y - y_0 + x_0(t-t_0)| \le r^3 x_0\right\}$$

If u *is a positive solution of* $\mathscr{L} u = 0$ *in* $H_r(z_0)$, *then*

$$u(z) \le M\, u(z_0)$$

for every $z \in S_{\theta r}(z_0)$. *The two constants* $\theta \in]0,1[$ *and* $M > 0$ *only depend on the operator* $\mathscr{L}$.



Harnack inequality for Kolmogorov operators

We achieve the proof by ***localizing*** the Harnack inequality proven for Kolmogorov operators by Di Francesco-Polidoro (2006):

Let Ω be an open subset of $\mathbb{R}^3$. Consider the following operator

$$\mathcal{K}v = \widetilde{a}(x,y,t)\partial_{xx}v + \widetilde{b}(x,y,t)\partial_x v + x\partial_y v - \partial_t v, \quad (x,y,t) \in \Omega.$$

Assume that $\widetilde{a}$ and $\widetilde{b}$ are bounded continuous (a bounded also from below) satisfying the following Hölder continuity condition

$$|\widetilde{a}(x,y,t) - \widetilde{a}(\xi,\eta,\tau)| \leq \widetilde{M}\left(|x-\xi| + |y-\eta+\xi(t-\tau)|^{1/3} + |t-\tau|^{1/2}\right)^{\alpha},$$

Fix $z_0 = (1,0,0)$, $r \in]0,1/2]$ and consider $H_r(1,0,0)$. Then, there exist two positive constants θ and M, only depending on the operator $\mathcal{K}$, such that for every non-negative solution v of $\mathcal{K}v = 0$ in Ω

$$v(z) \leq M\, v(1,0,0), \qquad \text{for every } z \in S_{\theta r}(1,0,0).$$

Paraboloid type Harnack inequality

As a corollary, we obtain the following

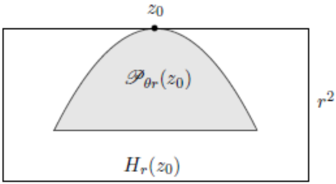
Corollary

Let $z_0 \in \mathbb{R}^+ \times \mathbb{R}^2$ and $r \in]0, 1/2]$. If u is a non negative solution of $\mathscr{L} u = 0$ in $H_r(z_0)$, then:

$$u(z) \leq M\, u(z_0)$$

for every z in the set

$$\mathscr{P}_{\theta r}(z_0) = \Big\{(x, y, t) \in \mathbb{R}^3 : 0 < t_0 - t \leq \theta^2 r^2, |x - x_0| \leq (t_0 - t)^{\frac{1}{2}} x_0, \\ |y - y_0 - (t_0 - t)x_0| \leq (t_0 - t)^{\frac{3}{2}} x_0\Big\}.$$



Admissible paths

Let fix (x, y, t), $(x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R} \times (0, T)$, where $t > t_0$. Reminding that

$$X_1 \sim (x, 0, 0), \quad X_0 \sim (0, x, 0), \quad -\partial_t \sim (0, 0, -1),$$

we construct the $\mathscr{L}$-admissible path $\gamma(s) = (x(s), y(s), t(s))$ (steering (x, y, t) to (x_0, y_0, t_0)) as the solution of the Cauchy problem

$$\begin{cases} \dot{x}(s) = \omega(s)x(s) & x(0) = x, \quad x(t - t_0) = x_0, \\ \dot{y}(s) = x(s) & y(0) = y, \quad y(t - t_0) = y_0 \\ \dot{t}(s) = -1, & t(0) = t, \quad t(t - t_0) = t_0 \end{cases}$$

where $\omega \in L^1([0, t - t_0])$. The function ω is said *control*.

Non-local Harnack inequality

Theorem (C.,Polidoro, Rossi)

There exist four positive constants θ, h, β and M, with $\theta < 1$ and $M > 0$, only depending on the operator $\mathscr{L}$ such that:

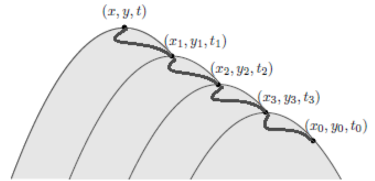
Let $T_0 < t_0 < t < T_1$ be fixed. Fix (x, y, t) and let $\omega \in L^1([0, t - t_0], \mathbb{R})$ be a control, with $\gamma : [0, t - t_0] \to \mathbb{R}^3$ the corresponding $\mathscr{L}$-admissible path starting from (x, y, t). Denote by $(x_0, y_0, t_0) = \gamma(t - t_0)$ its end-point.
Then, for every non negative solution $u : \mathbb{R}^+ \times \mathbb{R} \times (T_0, T_1)$ of $\mathscr{L} u = 0$ it holds

$$u(x_0, y_0, t_0) \le \left(\frac{t_0 - T_0}{t - T_0}\right)^{\beta} M^{1 + \frac{\Phi(\omega)}{h} + \frac{4(t - t_0)}{\theta^2}} u(x, y, t),$$

where

$$\Phi(\omega) = \int_0^{t - t_0} \omega^2(s)\, ds.$$

Idea of the proof



Step 1) We first show that we have

$$\gamma(t) \in \mathscr{P}_{\theta r}(x, y, t) \quad \text{for every } s \in [0, t - t_0],$$

where $r = \sqrt{t - T_0}$, if $t - T_0 \leq \frac{1}{4}$; $t - t_0 \leq \theta^2(t - T_0)$; and $\Phi(\omega) \leq 4 \log^2(3/2)$.
Step 2) Otherwise, define the sequence of times
$t_0 < t_k < t_{k-1} < \ldots < t_2 < t_1 < t$, according to the recurrence formula:

$$t_{j+1} = \max\left\{t_0,\ t_j - \theta^2/4, t_j - \theta^2(t_j - T_0),\ \inf\left\{s \ \text{ s.t. } \int_0^{t_j - s} |\omega(\tau)|^2 \, d\tau \leq h\right\}\right\}.$$

Set $r_j = \sqrt{t_j - t_{j+1}}/\theta$, and apply Step 1 on the $k+1$ intervals $[t_{j+1}, t_j]$. It holds

$$u(x_0, t_0) \leq Cu(x_k, t_k) \leq C^2 u(x_{k-1}, t_{k-1}) \leq \cdots \leq C^{k+1} u(x, t).$$

The Optimal Control Problem

We consider a path starting from $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R}^2$, and ending to $(x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$, with $t > t_0$. We formulate the following **Optimal Control Problem**

$$\Psi(x, y, t; x_0, y_0, t_0) = \min_{\omega \in L^1([0, t-t_0])} \int_0^{t-t_0} \omega^2(\tau) d\tau,$$

subject to constraint

$$\begin{cases} \dot{x}(s) = \omega(s)x(s), & x(0) = x, & x(t-t_0) = x_0, \\ \dot{y}(s) = x(s), & y(0) = y, & y(t-t_0) = y_0. \\ \dot{t}(s) = -1, & t(0) = t, & t(t-t_0) = t_0. \end{cases}$$

Such Problem can be explicitly solved by applying the Pontryagin Maximum Principle.

Quantitative information on Ψ

We summarize here some of the quantitative information about Ψ, that are written in terms of the function g defined as follows

$$g(r) = \begin{cases} \frac{\sinh(\sqrt{r})}{\sqrt{r}}, & r > 0, \\ 1, & r = 0, \\ \frac{\sin(\sqrt{-r})}{\sqrt{-r}}, & -\pi^2 < r < 0. \end{cases}$$

Theorem (C.,Polidoro, Rossi)

For every $(x, y, t), (x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$, with $t_0 < t$ and $y_0 > y$, we have

$$\begin{cases} \Psi(x, y, t; x_0, y_0, t_0) = E(t - t_0) + \frac{4(x+x_0)}{y_0 - y} - 4\sqrt{E + \frac{4xx_0}{(y_0-y)^2}}, \\ \text{if } E \geq -\frac{\pi^2}{t-t_0}; \\ \Psi(x, y, t; x_0, y_0, t_0) = E(t - t_0) + \frac{4(x+x_0)}{y_0 - y} + 4\sqrt{E + \frac{4xx_0}{(y_0-y)^2}}, \\ \text{if } -\frac{4\pi^2}{t-t_0} < E < -\frac{\pi^2}{t-t_0}. \end{cases}$$

where $E = E(x, y, t; x_0, y_0, t_0) = \frac{4}{(t-t_0)^2} g^{-1}\left(\frac{y_0 - y}{(t-t_0)\sqrt{xx_0}}\right).$

Asymptotic behaviours of the Value Function

Theorem (C.,Polidoro,Rossi)

For every $(x, y, t), (x_0, y_0, t_0) \in \mathbb{R} \times \mathbb{R}^+ \times (0, T)$ *the value function* $\Psi(x, y, t; x_0, y_0, t_0)$ *satisfies the following properties:*

1. $\Psi(x, y, t; x_0, y_0, t_0) = 0, \quad y < y_0, \; t > t_0;$
2. *Translation invariance and Scaling Property:*
   $\Psi(x, y, t; x_0, y_0, t_0) = \Psi\left(\frac{x}{x_0}, \frac{y-y_0}{x_0}, t - t_0, 1, 0, 0\right);$
   $\Psi(x, y, t; x_0, y_0, t_0) = \frac{1}{r}\Psi\left(x, \frac{y}{r}, \frac{t}{r}; x_0, \frac{y_0}{r}, \frac{t_0}{r}\right), \; r > 0;$
3. *Asymptotic Behaviours:*

$$\Psi(x, y, t; x_0, y_0, t_0) \approx \frac{4}{(t-t_0)} \log^2\left(\frac{y_0-y}{(t-t_0)\sqrt{xx_0}}\right) + \frac{4(x_0+x)}{y_0-y}, \quad as \; \frac{y_0-y}{(t-t_0)\sqrt{x_0 x}} \to +\infty$$

$$\Psi(x, y, t; x_0, y_0, t_0) \approx \frac{4(\sqrt{x}+\sqrt{x_0})^2}{y_0-y} - \frac{4\pi^2}{(t-t_0)}, \quad as \; \frac{y_0-y}{(t-t_0)\sqrt{x_0 x}} \to 0.$$

Lower bound for the fundamental solution

Lemma (Diagonal Estimates)

There exists a positive constant κ such that

$$\Gamma(1, -t, t; 1, 0, 0) \geq \frac{\kappa}{t^2}, \quad \textit{for every } t \in]0, 1/4].$$

Apply the non-local Harnack inequality by picking $\Gamma(x, y, t; 1, 0, 0)$ and $\Gamma(1, -\varepsilon t, \varepsilon t; 1, 0, 0)$:

Theorem (C.,Polidoro, Rossi)

Let $0 < \varepsilon < \frac{1}{4T}$ be fixed arbitrarily. There exists a positive constant $c^-_{\varepsilon,T}$ only depending on the operator $\mathscr{L}$, on ε and on T such that for every $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times]0, T]$ with $y < -\varepsilon t$ it holds

$$\Gamma(x, y, t; 1, 0, 0) \geq \frac{c^-_{\varepsilon,T}}{t^2} \exp\left(-C\Psi(x, y + \varepsilon t, t - \varepsilon t; 1, 0, 0)\right).$$

Upper Bound

Theorem (C.,Polidoro,Rossi)

Let T_0, T_1 *be fixed and consider the set* $\mathbb{R}^+ \times \mathbb{R} \times]T_0, T_1[$ *and let* $\Gamma(x, y, t; 1, 0, 0)$ *the fundamental solution of the operator* $\mathscr{L}$*. Denoting by* M_1 *the* L^∞*-norm of* $a(x, y, t)$ *and* $T = T_1 - T_0$*, then for every positive* ε*, there exists a positive constant* C_ε^+*, depending on* $\mathscr{L}$*, on* ε*,* T *and on the* L^∞*-norm of* $a(x, y, t)$ *such that*

$$\Gamma(x, y, t; 1, 0, 0) \leq \frac{C_\varepsilon^+}{t^2} \exp\left(-\tfrac{1}{16 M_1} \Psi(x, y - \varepsilon, t + \varepsilon; 1, 0, 0)\right)$$

for every $(x, y, t) \in \mathbb{R}^+ \times] - \infty, 0[\times (T_0, T_1)$*.*

Key tools

- Localize the Moser iteration proven by Pascucci-Cinti-Polidoro [2007] for non negative solution of Kolmogorov Operators in $\mathbb{R}^3$:

  Let (x_0, y_0, t_0) be any point of $\mathbb{R}^+ \times \mathbb{R}^2$, and let r, ρ with $0 < r/2 \leq \rho < r \leq 1/2$. Let $u \in L^p(H_r(x_0, y_0, t_0))$ be a non-negative weak solution of $\mathscr{L} u(x, y, t) = 0$ in $H_r(x_0, y_0, t_0)$, with $p \geq 1$. Then

$$\sup_{H_\rho(x_0,y_0,t_0)} u^p \leq \frac{\bar{c}}{(r-\rho)^6} \int_{H_r(x_0,y_0,t_0)} u^p.$$

- Use the Nash type estimates for the fundamental solution

  i) $\Gamma(x, y, t; x_0, y_0, t_0) \leq \frac{C_T}{(t-t_0)^2}$;

  ii) $\int_{\mathbb{R}^+\times\mathbb{R}} \Gamma^2(x, y, t; x_0, y_0, t_0) dx_0 dy_0 \leq \frac{C_T}{(t-t_0)^2}$;

Key tools: H-J-B equation

- Reminding that $\mathscr{L}_0 = X^2 + Y$ where:

$$X = x\partial_x, \quad Y = x\partial_y - \partial_t;$$

use the fact that the Value function Ψ satisfies the Hamilton-Jacobi-Bellman equation

$$Y\Psi(x,y,t;1,0,0) + \frac{1}{4}\Big(X\Psi(x,y,t;1,0,0)\Big)^2 = 0, \qquad (3)$$

and compare this information with

$$\mathscr{L}_0 u(x,y,t) = X^2 u(x,y,t) + Yu(x,y,t),$$

by choosing $u(x,y,t) = \exp\big(-k\Psi(x,y,t;1,0,0)\big)$ for a suitable positive constant k.

THANK YOU FOR YOUR ATTENTION!

Appendix: Yor result

The fundamental solution $\Gamma_0(x, y, t)$ of the operator $\mathscr{L}_0$ written by Yor in (1992) is

$$\Gamma_0(x, y, t) = \frac{e^{\frac{\pi^2}{t}}}{2\pi\sqrt{\pi t}} \cdot \frac{\sqrt{x}}{y^2} \exp\left(\tfrac{x+1}{2y}\right)\psi\Big(-\tfrac{\sqrt{x}}{y}, \tfrac{t}{2}\Big)$$

where

$$\psi\left(\frac{x}{y}, \tau\right) = \int_0^{\infty} e^{-\frac{\xi^2}{2\tau}} e^{-\frac{x}{y}\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{\tau}\right) d\xi.$$