Forward-Backward SDEs with distributional coefficients

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Kolmogorov-Fokker-Planck Equations: theoretical issues and applications

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This talk is based on:

- Issoglio E., Jing S. *Forward-backward SDEs with distributional coefficients* - preprint 2016 (arXiv:1605.01558)

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Introduction: Rough PDEs

$$(\text{PDE}) \quad \begin{cases} u_t(t,x) + L^b u(t,x) + f(t,x,u(t,x),\nabla u(t,x)) = 0, \\ u(T,x) = \Phi(x), \\ \forall (t,x) \in [0,T] \times \mathbb{R}^d, \end{cases}$$

where

- $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the unknown
- $L^b u_i = \frac{1}{2}\Delta u_i + b \cdot \nabla u_i$ is defined component by component
- $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}^d$
- $\beta \in (0, \frac{1}{2})$, $q \in \left(\frac{d}{1-\beta}, \frac{d}{\beta}\right)$
- b is a Schwartz distribution $b \in L^\infty([0,T]; H_q^{-\beta}(\mathbb{R}^d;\mathbb{R}^d))$

Motivations and Applications of rough PDEs

- PDEs like this with irregular fields b have been considered as models of transport of passive scalars in turbulent fluids, like the Kraichnan model[1]
- Recently the Kraichnan model has been investigated by physicists when the velocity field is a stochastic process [2][3]
- Example of rough drift: $b = \nabla B^H$ cut outside a domain, with B^H fractional Brownian noise with $H > \frac{1}{2}$
- In general, take the formal gradient of the realisation of some Hölder continuous random field, not necessarily Gaussian

[1] R. H. Kraichnan, Small scale structure of a scalar field convected by turbulence. *Physics of Fluids*, 11(5):945–953, 1968

[2] K. Gawedzki, Stochastic processes in turbulent transport. arXiv:0806.1949, 2008

[3] C. Pagani, Functional renormalization group approach to the Kraichnan model. *Phys. Rev. E*, 2015

Forward-backward SDE

- For b smooth, the solution of the PDE above can be expressed using a forward-backward SDE system via the well-known non-linear Feynman-Kac representation formula.

The forward-backward SDE system associated with (PDE) is

$$
\text{(FBSDE)} \quad \begin{cases} X_s^{t,x} = x + \int_t^s \mathrm{d}W_r, \\ Y_s^{t,x} = \Phi(X_T^{t,x}) + \int_s^T Z_r^{t,x} b(r, X_r^{t,x}) \mathrm{d}r, \\ \qquad + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \mathrm{d}r - \int_s^T Z_r^{t,x} \mathrm{d}W_r \\ \forall s \in [t, T]. \end{cases}
$$

- Then we have $u(s, X_s^{t,x}) = Y_s^{t,x}$ and $\nabla u(s, X_s^{t,x}) = Z_s^{t,x}$

Another Forward-backward SDE

It turns out one can associate another forward-backward SDE system with (PDE), that is

$$(\text{FBSDE*}) \qquad \begin{cases} X_s^{t,x} = \ x + \int_t^s b(r, X_r^{t,x})\mathrm{d}r + \int_t^s \mathrm{d}W_r, \\ Y_s^{t,x} = \Phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x})\mathrm{d}r, \\ \qquad - \int_s^T Z_r^{t,x}\mathrm{d}W_r \\ \forall s \in [t, T], \end{cases}$$

- (FBSDE) and (FBSDE*) are related by a Girsanov change of measure
- (FBSDE*) is associated with $L^b = \frac{1}{2}\Delta + b \cdot \nabla u$ in the PDE
- (FBSDE) is associated with $\frac{1}{2}\Delta$ in the PDE

Comments for the case b rough:

- The transformation above is justified if the coefficient b is not a distribution and is smooth enough
- Link not established yet, we study (FBSDE) and (FBSDE*) independently
- The associated PDE is the same for both systems!
- The connection between (PDE) and forward-backward equations (FBSDE) and (FBSDE*) that we will establish provides *new stochastic tools to represent and study turbulent PDEs*

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Aim of this talk:

- Find a unique mild solution of (PDE)
- Define a virtual-strong solution of (FBSDE)
- Find a unique virtual-strong solution of (FBSDE)
- Feynman-Kac formula: show link between the *mild solution* of (PDE) and the *virtual-strong solution* to (FBSDE)

Introduction

Rough PDE

Rough FBSDE

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PDE - I

Let us recall the semi-linear PDE we consider

$$\begin{cases} u_t(t,x) + \frac{1}{2}\Delta u + \nabla u \cdot b + f(t,x,u(t,x),\nabla u(t,x)) = 0, \\ u(T,x) = \Phi(x), \\ \forall (t,x) \in [0,T] \times \mathbb{R}^d, \end{cases}$$

Definition: The *mild solution* is given by

$$u(t) = P(T-t)\Phi + \int_t^T P(r-t)(\nabla u(r) \cdot b(r))\,dr + \int_t^T P(r-t) f(r, u(r), \nabla u(r))\,dr$$

where $P(t)$ is the heat semigroup generated by $\frac{1}{2}\Delta$

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PDE - II

Question: *What is the meaning of the product $\nabla u(r) \cdot b(r)$?*

- Use the notion of pointwise product: for $f, g \in \mathcal{S}'$ we define the product *if the limit exists in $\mathcal{S}'$*

$$fg := \lim_{j \to \infty} S^j f \, S^j g$$

- $S^j f(x) := \left(\rho\left(\frac{\cdot}{2^j}\right) \hat{f} \right)^{\vee}(x)$
- ρ is a mollifier with compact support

Fractional Sobolev spaces on $\mathbb{R}^d$

- H_p^α denotes the fractional Sobolev Space on L_p of order $\alpha \in \mathbb{R}$
- $H_p^\alpha \subset \mathcal{S}'$ ($\alpha < 0$: distributions; $\alpha \geq 0$: functions)

PDE - III

The pointwise product

- $b \in H_q^{-\beta}$ distribution, $0 < \beta < \delta$
- $\nabla u \in H_p^{\delta}$ function, $q > p \vee \frac{d}{\delta}$

Then $\nabla u \cdot b \in H_p^{-\beta}$ and

$$\|\nabla u \cdot b\|_{H_p^{-\beta}} \le c \, \|b\|_{H_q^{-\beta}} \, \|\nabla u\|_{H_p^{\delta}}$$

Remark: the product $b \cdot \nabla u$ is a distribution

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PDE - IV

- look for a fixed point $u = I(u)$
- solution u is a weakly differentiable function $u \in H_p^{1+\delta}$
- drift is a distribution $b \in H_q^{-\beta}$
- $0 < \beta < \delta < \frac{1}{2}$
- $f(r,\cdot,\cdot)$ is Lipschitz

$$u(t) = P(T-t)\Phi + \int_t^T P(r-t) \underbrace{\Big(\overbrace{\nabla u(r)}^{\in H_p^{\delta}} \cdot \overbrace{b(r)}^{\in H_q^{-\beta}} \Big)}_{\in H_p^{-\beta}} dr + \int_t^T P(r-t) \underbrace{f(r, u(r), \nabla u(r))}_{\in H_p^{1+\delta}} dr$$

- the semigroup lifts almost 2 derivatives: from $-\beta$ to $1+\delta$

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PDE - V

Theorem

- $b \in L^\infty(0, T;\, H_q^{-\beta})$
- $\Phi \in H_p^{1+\delta+2\gamma}$
- $f : [0, T] \times H_p^{1+\delta} \times H_p^{\delta} \to H_p^0$ be such that $\sup_t \|f(t,0,0)\|_{H_p^0} \leq C$ and

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\|_{H_p^0} \leq L(\|u_1 - u_2\|_{H_p^{1+\delta}} + \|v_1 - v_2\|_{H_p^{\delta}})$$

Then there exists a unique mild solution $u \in C([0, T], H_p^{1+\delta})$ to (PDE).
Moreover $u \in C^\gamma([0, T], C^{1,\alpha})$ for some $\gamma, \alpha > 0$ small enough.

Introduction

Rough PDE

Rough FBSDE

Existing Literature on FBSDEs

- On strong solutions: seminal work of Pardoux&Peng 1990, Pardoux&Peng 1992, Antonelli 1993.
  Many other authors...
- On weak solutions: Antonelli&Ma 2003, Buckdahn&Engelbert&Rascanu 2004, Lejay 2004, Delarue&Guatteri 2006, Ma&Zhang&Zheng 2008
- With distributional coefficients: Russo&Wurzer 2015

FBSDE - I

Let us recall the FBSDE we consider:

$$
\begin{cases}
X_s^{t,x} = & x + \int_t^s \mathrm{d}W_r, \\
Y_s^{t,x} = & \Phi(X_T^{t,x}) - \int_s^T Z_r^{t,x} \mathrm{d}W_r + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \mathrm{d}r \\
& + \int_s^T Z_r^{t,x} b(r, X_r^{t,x}) \mathrm{d}r, \\
\forall s \in [t, T].
\end{cases}
$$

- $X_s^{t,x}$ is a Brownian motion on (Ω, F, P) starting in x at time t
- $\mathbb{F}$ is the filtration generated by W (or X equivalently)
- $\int_s^T Z_r^{t,x} b(r, X_r^{t,x}) \mathrm{d}r$ is not well-defined a priori

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FBSDE - II

- Auxiliary PDE (u is the mild solution of (PDE))

$$(1) \qquad \begin{cases} w_t + \frac{1}{2}\Delta w = \nabla u \cdot b, \\ w(T,x) = 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \end{cases}$$

- Itô trick: If b was smooth, Itô's formula for $w(\cdot, X)$ would give $0 - w(s, X_s) = \int_s^T \nabla w(r, X_r) \mathrm{d}W_r + \int_s^T \nabla u(r, X_r) b(r, X_r) \mathrm{d}r$
- and we would have $(Y, Z) = (u(\cdot, X), \nabla u(\cdot, X))$
- hence $\int_s^T Z_r b(r, X_r) \mathrm{d}r = -w(s, X_s) - \int_s^T \nabla w(r, X_r) \mathrm{d}W_r$

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FBSDE - II

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- Itô trick: If b was smooth, Itô's formula for $w(\cdot, X)$ would give $0 - w(s, X_s) = \int_s^T \nabla w(r, X_r) \mathrm{d}W_r + \int_s^T \nabla u(r, X_r) b(r, X_r) \mathrm{d}r$
- and we would have $(Y, Z) = (u(\cdot, X), \nabla u(\cdot, X))$
- hence $\int_s^T Z_r b(r, X_r) \mathrm{d}r = -w(s, X_s) - \int_s^T \nabla w(r, X_r) \mathrm{d}W_r$

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FBSDE - III

Definition

A *virtual-strong solution* to the backward SDE in (FBSDE) is a couple (Y, Z) such that

- Y is continuous and $\mathbb{F}$-adapted and Z is $\mathbb{F}$-progressively measurable;
- $E\left[\sup_{r\in[t,T]} |Y_r|^2\right] < \infty$ and $E\left[\int_t^T |Z_r|^2 \mathrm{d}r\right] < \infty$;
- for all $s \in [t, T]$, the couple satisfies the following backward SDE P-almost surely

$$Y_s = \Phi(X_T) - \int_s^T Z_r \mathrm{d}W_r + \int_s^T f(r, X_r, Y_r, Z_r)\mathrm{d}r - w(s, X_s) - \int_s^T \nabla w(r, X_r)\mathrm{d}W_r. \tag{2}$$

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FBSDE - IV

- every term in the backward SDE (2) is now well defined
- SDE (2) is not written in a classical form
- to find a virtual solution we transform (2) as follows:
  •$\hat{Y}_s := Y_s + w(s, X_s)$;
  •$\hat{Z}_s := Z_s + \nabla w(s, X_s)$;
  •$\hat{f}(s, x, y, z) := f(s, x, y - w(s, x), z - \nabla w(s, x))$

We get the following *auxiliary backward SDE*

$$(3) \qquad \hat{Y}_s = \Phi(X_T) - \int_s^T \hat{Z}_r \mathrm{d}W_r + \int_s^T \hat{f}(r, X_r, \hat{Y}_r, \hat{Z}_r)\mathrm{d}r$$

FBSDE - V

Proposition

(i) If (Y, Z) is a virtual-strong solution of (FBSDE), then

$$(\hat{Y}, \hat{Z}) := (Y + w(\cdot, X), Z + \nabla w(\cdot, X))$$

is a solution of the auxiliary backward SDE (3)

(ii) If $(\hat{Y}, \hat{Z})$ is a solution of the auxiliary backward SDE (3), then

$$(Y, Z) := (\hat{Y} - w(\cdot, X), \hat{Z} - \nabla w(\cdot, X))$$

is a virtual-strong solution of (FBSDE)

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FBSDE - VI

Assumptions

- $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and bounded;
- $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \to \mathbb{R}^d$ with

$$|f(t, x, y, z) - f(t, x, y', z')| \leq L(|y - y'| + |z - z'|)$$

  uniformly in t and x;
- $\sup_{t,x} |f(t, x, 0, 0)| \leq C$ and $\sup_t \|f(t, \cdot, 0, 0)\|_{H_p^0} \leq C$.

Proposition

$\exists!$ strong solution $(\hat{Y}, \hat{Z})$ to the auxiliary backward SDE (3).

Corollary

$\exists!$ virtual-strong solution (Y, Z) to (FBSDE).

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A Feynman-Kac formula

Theorem

Let u be the unique mild solution of (PDE) and X be a Brownian motion such that $X_t = x$.
Then the couple $(u(\cdot, X), \nabla u(\cdot, X))$ is a virtual-strong solution of (FBSDE).

On the other hand, let $(Y^{t,x}, Z^{t,x})$ be the unique virtual-strong solution of (FBSDE) and suppose that $Y_s^{t,x} = \alpha(s, X_s^{t,x})$ and $Z_s^{t,x} = \beta(s, X_s^{t,x})$ for some deterministic functions α, β with appropriate regularity. Then the unique mild solution of (PDE) can be written as $u(t,x) = Y_t^{t,x}$ and moreover we have that $\nabla u(t,x) = Z_t^{t,x}$.

Thank You for Your Attention.

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