# RANDOM FLIGHTS: POISSONIAN ENVIRONMENT, DIFFUSION APPROXIMATIONS, LOCAL LIMIT THEOREMS

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Modena, 2017

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International Workshop "Kolmogorov-Fokker-Planck Equations : theoretical issues and applications"

(April 10 - April 11 , 2017, Modena)

# Introduction.

We consider the moving particle process in $R^d$ which is defined in the following way. There are two independent sequences $(T_k)$ and $(\varepsilon_k)$ of random variables.

The variables $T_k$ are non negative and $\forall k \ \ T_k \leq T_{k+1}$, while variables $\varepsilon_k$ form an i.i.d sequence with common distribution concentrated on the unit sphere $S^{d-1}$.

The values $\varepsilon_k$ are interpreted as the directions, and $T_k$ as the moments of change of directions.

# Introduction.

A particle starts from zero and moves in the direction $\varepsilon_1$ up to the moment $T_1$. It then changes direction to $\varepsilon_2$ and moves on within the time interval $T_2 - T_1$, etc. The speed is constant at all sites. The position of the particle at time $t$ is denoted by $X(t)$.

# Introduction.

Study of the processes of this type has a long history. The first work dates back probably to Pearson (1905) and continued by Kluyer (1906) and Rayleigh (1919). Mandelbrot (1982) considered the case where the increments $T_n - T_{n-1}$ form i.i.d. sequence with the common law having a heavy tail. He also introduced the term "Levy flights" later evolved into the "Random flights".

To date, a large number of works were accumulated, devoted to the study of such processes, we mention here only articles of A.Kolesnik (2009), E. Orsingher and A. De Gregorio (2012, 2015) and E. Orsingher and R. Garra (2014) which contain an extensive bibliography and where for different assumptions on $(T_k)$ and $(\varepsilon_k)$ the exact formulas for the distribution of $X(t)$ were derived.

# Introduction.

Our goals are different.

Firstly, we are interested in the global behavior of the process $X = \{X(t),\ t \in R_+\}$, namely, we are looking for conditions under which the processes $\{Y_T,\ T > 0\}$,

$$Y_T(t) = \frac{1}{B(T)} X(tT), \quad t \in [0,1],$$

weakly converges in $C[0,1]$ : $Y_T \Longrightarrow Y,\ B_T \longrightarrow \infty,\ T \longrightarrow \infty.$

Secondly, we want to construct diffusion approximations for the process $X(t)$ and evaluate the accuracy of such approximations.

# Introduction.

From now on we suppose that the points $(T_k)$, $T_k \leq T_{k+1}$, form a Poisson point process in $R_+$.

It is clear that in the homogeneous case the process $X(t)$ is a conventional random walk because the spacings $T_{k+1} - T_k$ are independent, and then the limit process is Brownian motion.

In the non homogeneous case the situation is more complicated as these spacings are not independent. Nevertheless it was possible to distinguish three modes that determine different types of limiting processes.

For a more precise description of the results it is convenient to assume that $T_k = f(\Gamma_k)$, where f is certain function and $(\Gamma_k)$ is a standard homogeneous Poisson point process on $R_+$.

# Introduction.

If the function $f$ has power growth,

$$f(t) = t^{\alpha},\ \alpha > 1/2,$$

the behavior of the process is analogous to the uniform case and then in the limit we obtain a Gaussian process which is a lineally transformed Brownian motion

$$Y(t) = \int_0^t K_{\alpha}(s) dW(s),$$

where $W$ is a process of Brownian motion, for which the covariance matrix coincides with the covariance matrix of $\varepsilon_1$.

# Introduction.

In the case of exponential growth,

$$f(t) = e^{t\beta},\ \beta > 0,$$

the limiting process is piecewise linear with an infinite number of units, but $\forall \epsilon > 0$ the number of units in the interval $[\varepsilon, 1]$ will be a.s. finite.

# Introduction.

Finally, with the super exponential growth of $f$, the process degenerates : its trajectories are linear functions :

$$Y(t) = \varepsilon t, \quad t \in [0,1], \quad \varepsilon \overset{Law}{=} \varepsilon_1.$$

In the second part the process $X(t)$ is considered as a Markov chain. We construct diffusion approximations for this process and investigate their accuracy. The main tool in this part is the parametrix method (K., 2012).

# Results, Th.1, 1).

Remind that we suppose $T_k = f(\Gamma_k)$, where $(\Gamma_k)$ is a standard homogeneous Poisson point process on $R_+$. Assume also that $E\varepsilon_1 = 0$.

It is more convenient consider at first behavior of processes

$$Z_n(t) = Y_{T_n}(t),$$

as for $T = T_n$ the paths of $Z_n$ have an integer number of full segments on the interval [0,1].

**Theorem 1**

*Under previous assumptions*

1) *If the function $f$ has power growth : $f(t) = t^\alpha$, $\alpha > 1/2$, we take $B(T) = T^{\frac{2\alpha-1}{2\alpha}}$.*
   *Then $Z_n \Longrightarrow Y$, where $Y$ is a Gaussian process*

$$Y(t) = \sqrt{2\alpha} \int_0^t s^{\frac{\alpha-1}{2\alpha}} dW(s),$$

   *and $W$ is a process of Brownian motion, for which the covariance matrix of $W(1)$ coincides with the covariance matrix of $\varepsilon_1$.*

# Results, Th.1, 2)

**Theorem 1**

2) *If the function $f$ has exponential growth : $f(t) = e^{t\beta}$, $\beta > 0$, we take $B(T) = T$.*

*Then $Z_n \Longrightarrow Y$, where $Y$ is a continuous picewise lineal process with the vertices at the points $(t_k, Y(t_k))$,*

$$t_k = e^{-\beta \Gamma_{k-1}}, \quad \Gamma_0 = 0,$$

$$Y(t_k) = \sum_{i=k}^{\infty} \varepsilon_k (e^{-\beta \Gamma_{i-1}} - e^{-\beta \Gamma_i}), \quad Y(0) = 0.$$

# Theorem 1, Th.1, 3).

**Theorem 1**

3) In super exponential case suppose that $f$ is increasing absolutely continuous and such that

$$\lim_{t\to\infty} \frac{f'(t)}{f(t)} = +\infty.$$

We take $B(T) = T$.

Then $\frac{T_n}{T_{n+1}} \to 0$ in probability, and $Z_n \Longrightarrow Y$, where the limiting process $Y$ degenerates :

$$Y(t) = \varepsilon_1 t, \quad t \in [0, 1].$$

# Remark 1

**Remark**

*In the case of power growth the limiting process admits the following representation :*

$$Y(t) \overset{\mathcal{L}}{=} \alpha\sqrt{\frac{2}{2\alpha-1}}W(t^{\frac{2\alpha-1}{\alpha}}),$$

*where, as before, $W$ is a Brownian motion, for which the covariance matrix of $W(1)$ coincides with the covariance matrix of $\varepsilon_1$.*
*It is clear that we can also express $Y$ in another way :*

$$Y(t) \overset{\mathcal{L}}{=} \alpha\sqrt{\frac{2}{2\alpha-1}}K^{\frac{1}{2}}w(t^{\frac{2\alpha-1}{\alpha}}),$$

*where $w$ is a standard Browniam motion and $K$ is the covariance matrix of $\varepsilon_1$.*

# Remark 2

## Remark

*In the case of exponential growth it is possible to describe the limiting process $Y$ in the following way :*
*We take a p.p.p.* $\mathbf{T} = (t_k)$, $t_k = e^{-\beta \Gamma_{k-1}}$, *defined on* $(0, 1]$, *and define a step process*
$\{Z(t),\ t \in (0, 1]\}$,

$$Z(t) = \varepsilon_k \quad \textit{for} \ \ t \in (t_{k+1}, t_k].$$

*Then*

$$Y(t) = \int_0^t Z(s)\,\mathrm{d}s.$$

# Comments.

Poof of Th.1 (First part). We have

$$t_{n,k} = \frac{T_k}{T_n} = \left(\frac{\Gamma_k}{\Gamma_n}\right)^{\alpha}, \quad B_n = n^{\frac{2\alpha-1}{2}}, \qquad Z_n(t_{n,k}) = \frac{1}{B_n}\sum_1^n \varepsilon_i(\Gamma_i^{\alpha} - \Gamma_{i-1}^{\alpha}).$$

1-st step. Compare $Z_n(\cdot)$ with $V_n(\cdot)$ where

$$V_n(t_{n,k}) = \frac{\alpha}{B_n}\sum_1^n \varepsilon_i \gamma_i \Gamma_{i-1}^{\alpha-1}, \qquad \gamma_i = \Gamma_i - \Gamma_{i-1},$$

and show that

$$\|Z_n - V_n\|_{\infty} \xrightarrow{P} 0.$$

# Comments

2-nd step.
Compare $V_n(\cdot)$ with $W_n(\cdot)$ where

$$W_n(t_{n,k}) = \frac{\alpha}{B_n} \sum_1^n \varepsilon_i \gamma_i (i-1)^{\alpha-1},$$

and show that

$$\|W_n - V_n\|_\infty \xrightarrow{P} 0.$$

# Comments

3-d step.
Show that the process

$$U_n\left(\left(\frac{k}{n}\right)^{\alpha}\right) = \frac{\alpha}{B_n}\sum_{1}^{n}\varepsilon_i\gamma_i(i-1)^{\alpha-1},$$

converges weakly to the process

$$Y(t) = \sqrt{2\alpha}\int_0^t s^{\frac{\alpha-1}{2\alpha}}\,dW(s).$$

Show finally that the convergence $W_n \Rightarrow Y$ follows from the convergence $U_n \Rightarrow Y$. ■

In the second part we consider a model of random flight which is equivalent to the study of random broken lines $\{X_n(t), t \in [0,1]\}$ with the vertices $(\frac{k}{n},\ X_n(\frac{k}{n}))$, and such that $(h = \frac{1}{n})$

$$X_n\left((k+1)h\right) = X_n(kh) + hb(X_n(kh)) + \sqrt{h}\xi_k(X(kh)),$$

$$X_n(0) = x_0, \quad \xi_k(X_n(kh)) = \rho_k \sigma(X_n(kh))\varepsilon_k, \tag{1}$$

where $\{\varepsilon_k\}$ and $\{\rho_k\}$ are two independent sequences such that

$\{\varepsilon_k\}$ are i.i.d. r. v. uniformly distributed on the unit sphere $S^{d-1}$,
$\{\rho_k\}$ are i.i.d. r. v. having a density, $\rho_k \geq 0$, $E\rho_k^2 = d$,

$b : R^d \longrightarrow R^d$ is a bounded measurable function and
$\sigma : R^d \longrightarrow R^d \times R^d$ is a bounded measurable matrix function.

# Results, Th.2.

**Theorem 2**

Let $X = \{X(t), t \in [0,1]\}$ be a solution of stochastic equation

$$X(t) = x_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dw(s). \qquad (2)$$

Suppose that $b$ and $\sigma$ are continuous functions satisfying Lipschits condition

$$|b(t) - b(s)| + |\sigma(t) - \sigma(s)| \leq K|t - s|.$$

Moreover it is supposed that $b(x)$ and $\frac{1}{\det(\sigma(x))}$ are bounded.
Then

$$X_n \Rightarrow X \quad \text{in} \quad \mathbb{C}[0,1].$$

# Comments

To prove the weak convergence we use the approach of Stroock and Varadhan (1979). Under our assumptions the diffusion coefficients $a$ and $b$ have the property that for each $x \in R^d$ the martingale problem for $a$ and $b$ has exactly one solution $P_x$ starting from $x$ (that is well posed). It remains to check the conditions from Stroock and Varadhan (1979) which imply the weak convergence of our sequence of Markov chains to this unique solution $P_x$ .

# Comments

Our next result is about approximation of transition density. We consider now more general models given by a triplet $(b(x), \sigma(x), f(r;\theta)),\ x \in R^d,\ r \geq 0,\ \theta \in R^+$, where $b(x)$ is a vector field, $\sigma(x)$ is a $d \times d$ matrix, $a(x) := \sigma\sigma^T(x) > \delta I,\ \delta > 0$, and $f(r;\theta)$ is a radial density depending on a parameter $\theta$ controlling the frequency of changes of directions, namely, the frequency increases when $\theta$ decreases. Suppose $X(0) = x_0$. The vector $b(x_0)$ acts shifting a particle from $x_0$ to $x_0 + \Delta(\theta)b(x_0)$, where $\Delta(\theta) = c_d\theta^2$, $c_d > 0$. Several examples of such functions $\Delta(\theta)$ for different models will be given below. Define

$$\mathcal{E}_{x_0}(r) := \{x : \left|a^{-1/2}(x_0)(x - x_0 - \Delta(\theta)b(x_0))\right|^2 = r^2\},$$

$$\mathcal{S}^d_{x_0}(r) := \{y : |y - x_0 - \Delta(\theta)b(x_0)|^2 = r^2\},$$

The initial direction is defined by a random variable $\xi_0$, the law of $\xi_0$ is a pushforward of the spherical measure on $\mathcal{S}^d_{x_0}(1)$ under affine change of variables

$$x - x_0 - \Delta(\theta)b(x_0) = a^{1/2}(x_0)(y - x_0 - \Delta(\theta)b(x_0))$$

# Comments

Then particle moves along the ray $l_{x_0}$ corresponding to the directional unit vector

$$\varepsilon_0 := \frac{\xi_0 - x_0 - \Delta(\theta)b(x_0)}{|\xi_0 - x_0 - \Delta(\theta)b(x_0)|},$$

and changes the direction in $(r, r + dr)$ with probability

$$|a^{-1/2}(x_0)\varepsilon_0| \cdot f(r \left| a^{-1/2}(x_0)\varepsilon_0 \right|)dr. \tag{3}$$

Let $\rho_0$ be a random variable independent on $\xi_0$ and distributed on $l_{x_0}$ with the radial density (3). We consider the point $x_1 = x_0 + \Delta(\theta)b(x_0) + \rho_0\varepsilon_0$. Let $(\varepsilon_k, \rho_k)$ be independent copies of $(\varepsilon_0, \rho_0)$. Starting from $x_1$ we repeat the previous construction to obtain $x_2 = x_1 + \Delta(\theta)b(x_1) + \rho_1\varepsilon_1$. After $n$ switching we get a point $x_n$,

$$x_n = x_{n-1} + \Delta(\theta)b(x_{n-1}) + \rho_{n-1}\varepsilon_{n-1}.$$

Now we make our main assumption about the radial density :
**(A1)** The funciton $f(r;\theta)$ is homogenious of degree $-1$, that is

$$f(\lambda r; \lambda\theta) = \lambda^{-1} f(r;\theta),\ \forall\lambda \neq 0.$$

Denote by $p_{\mathcal{E}}(n,x,y)$ the transition density after $n$ switching in the RF-model described above. To obtain the one step transition density $p_{\mathcal{E}}(1,x,y)$ (we write $(x,y)$ instead of $(x_0,x_1)$) we use the inverse Fourier transform, the one-step characteristic function and **(A1).** After easy calculations we get

$$p_{\mathcal{E}}(1,x,y) = \Delta^{-d/2}(\theta) q_x \left( \frac{y - x - \Delta(\theta) b(x)}{\sqrt{\Delta(\theta)}} \right), \tag{4}$$

where

$$q_x(z) = \frac{2^{\frac{d-2}{2}} \Gamma\left(\frac{d}{2}\right)}{(2\pi)^d} \int_{R^d} \cos\langle \tau, z\rangle \left[ \int_0^\infty \frac{J_{\frac{d-2}{2}}(\rho \left|a^{1/2}(x)\tau\right|)}{\left(\rho\left|a^{1/2}(x)\tau\right|\right)^{\frac{d-2}{2}}} f(\rho; c_d) d\rho \right] d\tau. \tag{5}$$

# Example 1

We put $\Delta(\theta) = (d+1)^2\theta^2$ and

$$f(r;\theta) = \frac{1}{\Gamma(d)} r^{-1} \left(\frac{r}{\theta}\right)^d \exp\left(-\frac{r}{\theta}\right).$$

Using formula 6.623 (2) on page 726 from Gradshtein and Ryzhik (1963), and the doubling formula for the Gamma function we obtain

$$p_{\mathcal{E}}(1,x,y) = \Delta^{-d/2}(\theta) q_x \left( \frac{y - x - \Delta(\theta) b(x)}{\sqrt{\Delta(\theta)}} \right),$$

where

$$q_x(z) = \frac{(d+1)^{d/2}}{2^d \pi^{(d-1)/2} \Gamma\left(\frac{d+1}{2}\right) \left|\det a^{1/2}(x)\right|} e^{-\sqrt{d+1}\left|a^{-1/2}(x) z\right|}.$$

It is easy to check that

$$\int z_i q_x(z) = 0, \quad \int z_i z_j q_x(z) dz = a_{ij}(x).$$

# Example 2

We put $\Delta(\theta) = \theta^2/2$ and

$$f(r;\theta) = C_d r^{-1} \left(\frac{r}{\theta}\right)^d \exp\left(-\frac{r^2}{\theta^2}\right),$$

where $C_d = \frac{2^{(d+1)/2}}{(d-2)!!\sqrt{\pi}}$ if $d$ is odd, and $C_d = \frac{2}{[(d-2)/2]!}$ if $d$ is even. From formula 6.631 (4) on page 731(Gradshtein and Ryzhik) we obtain

$$p_{\mathcal{E}}(1,x,y) = \Delta^{-d/2}(\theta)\phi_x\left(\frac{y-x-\Delta(\theta)b(x)}{\sqrt{\Delta(\theta)}}\right), \qquad (6)$$

where

$$\phi_x(z) = \frac{1}{(2\pi)^{d/2}\sqrt{\det a(x)}} \exp\left(-\frac{1}{2}\left\langle a^{-1}(x)z, z\right\rangle\right).$$

# Comments

It is easy to see that the transition density (4) corresponds to the one step transition density in the following Markov chain model

$$X_{(k+1)\Delta(\theta)} = X_{k\Delta(\theta)} + \Delta(\theta)\, b(X_{k\Delta(\theta)}) + \sqrt{\Delta(\theta)}\xi_{(k+1)\Delta(\theta)},$$

where the conditional density (under $X_{k\Delta(\theta)} = x$) of the innovations $\xi_{(k+1)\Delta(\theta)}$ is equal to $q_x(\cdot)$. If we put $\theta = \theta_n = \sqrt{\frac{2}{n}}$, then $\Delta(\theta_n) = \frac{1}{n}$ and we obtain a sequence of Markov chains defined on an equidistant grid

$$X_{\frac{k+1}{n}} = X_{\frac{k}{n}} + \frac{1}{n} b(X_{\frac{k}{n}}) + \frac{1}{\sqrt{n}}\xi_{\frac{k+1}{n}}, \quad X_0 = x_0. \tag{7}$$

Note that the triplet $(b(x), \sigma(x), f(r;\theta)),\ x \in R^d,\ r \geq 0,\ \theta \in R^+$, of the Example 2 corresponds to the classical Euler scheme for the $d$-dimensional SDE

$$dX(t) = b(X_t)dt + \sigma(X_t)dW(t), \quad X(0) = x_0. \tag{8}$$

# Theorem 3.

**(A2)** The function $a(x) = \sigma\sigma^T(x)$ is uniformly elliptic.

**(A3)** The functions $b(x)$ and $\sigma(x)$ and their derivatives up to the order six are continuous and bounded uniformly in $x$. The 6-th derivative is globally Lipschitz.

**Theorem**

*Under assumptions (A2), (A3) we have the following expansion for the model with one step transition density (6) : for any positive integer $S$ as $n \to \infty$*

$$\sup_{x,y \in R^d} \left(1 + |y - x|^S\right) \cdot |p_{\mathcal{E}}(n, x, y) - p(1, x, y) - \\ - \frac{1}{2n} p \otimes \left(L_*^2 - L^2\right) p(1, x, y)| = O(n^{-3/2}), \quad (9)$$

*where*

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \partial^2_{x_i x_j} + \sum_{i=1}^{d} b_i(x) \partial_{x_i}. \quad (10)$$

# Remark

The operator $L_*$ in (9) is the same operator as in (10) but with coefficients "frozen" ay $x$. Clearly, $L = L_*$ but, in general, $L^2 \neq L_*^2$. The convolution type binary operation $\otimes$ is defined for functions $f$ and $g$ in the following way

$$(f \otimes g)(t, x, y) = \int_0^t ds \int_{R^d} f(s, x, z) g(t - s, z, y) dz.$$

**Proof.** It follows immidiately from Theorem 1 of K. and Mammen (2009).

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