NASH ESTIMATES AND UPPER BOUNDS FOR NON-HOMOGENEOUS KOLMOGOROV EQUATIONS

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1 The Problem

We consider the Kolmogorov-type equation

$$Lu := \sum_{i,j=1}^{m_0} \partial_{x_i}(a_{ij}\partial_{x_j}u) + \sum_{i=1}^{m_0} \partial_{x_i}(a_i u) + cu + \sum_{i,j=1}^{d} b_{ij}x_j\partial_{x_i}u + \partial_t u = 0$$

where $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, $m_0 \leq d$, the matrix $B := (b_{ij})_{1\leq i,j\leq d}$ has constant real entries and the coefficients $a_{ij} = a_{ji}$, a_i, c for $1 \leq i,j \leq m_0$, are bounded, measurable functions.
The operator L has to be interpreted as a perturbation of

$$L_0 u := \frac{1}{2}\sum_{i=1}^{m_0} \partial_{x_i x_i}u + \sum_{i,j=1}^{d} b_{ij}x_j\partial_{x_i}u + \partial_t u$$

which we call the *principal part* of L. If we denote

$$Yu := \sum_{i,j=1}^{d} b_{ij} x_j \partial_{x_i} u + \partial_t u$$

then, according to Hörmander's theorem, L_0 is *hypoelliptic* if

$$\text{rank Lie}\left\{\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\right\}(t,x) = d+1, \qquad \text{for all } (t,x) \in \mathbb{R}^{d+1}.$$

It was proved in *Lanconelli and Polidoro (1994)* that for the operator L_0 the last condition is equivalent to what we assume as a standing assumption.

Assumption 1
The matrix $B := (b_{ij})_{1 \le i,j \le d}$ takes the block-form

$$B = \begin{pmatrix} * & * & \cdots & * & * \\ B_1 & * & \cdots & * & * \\ 0 & B_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_\nu & * \end{pmatrix}$$

where each B_i is a $(m_i \times m_{i-1})$-matrix of rank m_i with

$$m_0 \ge m_1 \ge \cdots \ge m_\nu \ge 1, \qquad \sum_{i=0}^{\nu} m_i = d,$$

and the blocks denoted by "$*$" are arbitrary.

Moreover, to preserve the sub-elliptic character of the operator L_0 we also require:

Assumption 2
There exists a positive constant μ such that

$$\mu^{-1}|\xi|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(t,x)\xi_i\xi_j \le \mu|\xi|^2, \qquad \xi \in \mathbb{R}^{m_0},\ (t,x) \in \mathbb{R}^{d+1}.$$

Our aim is to obtain a Gaussian upper bound for the fundamental solution $\Gamma = \Gamma(t,x;T,y)$ of L, which is independent of the smoothness of the coefficients. In fact, the constants appearing in our upper bound will only depend on the L^∞-norms of the coefficients and the matrix B. For this reason we introduce the following class.

Notation 1
Let $M > 0$ *and* B *a* $d \times d$ *matrix satisfying Assumption 1. We denote by* $\mathcal{K}_{M,B}$ *the class of Kolmogorov operators* L *that satisfy our assumptions with constant* μ *and norms* $\|a_{ij}\|_\infty$, $\|a_i\|_\infty$, $\|c\|_\infty$ *smaller than* M, *for all* $i, j = 1, ..., m_0$.

We are ready to state our main result.

Theorem 3
Let $L \in \mathcal{K}_{M,B}$ *and* $T_0 > 0$. *If* Γ *is a fundamental solution of the operator* L, *then there exists a positive constant* C, *only dependent on* M, B *and* T_0, *such that*

$$\Gamma(t, x; T, y) \leq \frac{C}{(T-t)^{\frac{Q}{2}}} \exp\left(-\frac{1}{C}\left|\mathcal{D}\left((T-t)^{-\frac{1}{2}}\right)\left(x - e^{-(T-t)B}y\right)\right|^2\right),$$

for $0 < T - t \leq T_0$ *and* $x, y \in \mathbb{R}^d$, *with*

$$\mathcal{D}(r) := \mathrm{diag}(rI_{m_0}, r^3 I_{m_1}, \ldots, r^{2\nu+1} I_{m_\nu}), \qquad r > 0,$$

where I_{m_i} *denotes the* $(m_i \times m_i)$*-identity matrix, and*

$$Q := m_0 + 3m_1 + \cdots + (2\nu + 1)m_\nu.$$

Remark 1
Theorem 3 is an a priori estimate in the sense that it is derived under conditions that do not guarantee the actual existence of a fundamental solution. In the case of Hölder continuous coefficients, the existence of a fundamental solution was proved by Polidoro (1994), *for homogeneous Kolmogorov equations* ($*$*-blocks of* B *are zero), and by* Di Francesco and Pascucci (2005), *for non-homogeneous Kolmogorov equations (the case consider in this talk).*

Remark 2
The exponent $\frac{Q}{2}$ *appearing in the estimate is optimal, as it can be easily seen in the case of constant-coefficient Kolmogorov operators (whose fundamental solution is explicit). Notice the difference with respect to the uniformly parabolic case: for instance, in* $\mathbb{R}^3$, *for the heat operator* $\partial_{x_1x_1} + \partial_{x_2x_2} + \partial_t$ *we have* $Q = 2$, *while for the prototype Kolmogorov operator* $\partial_{x_1x_1} + x_1\partial_{x_2} + \partial_t$ *we have* $Q = 4$.

Remark 3
The previous theorem generalizes the classical results by Nash (1958), Aronson (1967) *and* Davies (1987), *for uniformly parabolic equations, and* Pascucci and Polidoro (2003) *for Kolmogorov equations with null* $*$*-blocks in the matrix* B.

2 Geometric properties of the operator L_0

Constant-coefficient Kolmogorov operators are naturally associated to *linear* stochastic differential equations: indeed,

$$L_0 u = \frac{1}{2}\sum_{i=1}^{m_0} \partial_{x_i x_i} u + \sum_{i,j=1}^{d} b_{ij} x_j \partial_{x_i} u + \partial_t u$$

is the infinitesimal generator of the d-dimensional SDE

$$dX_T^{t,x} = BX_T^{t,x} dT + \sigma dW_T, \quad X_t^{t,x} = x \tag{2.1}$$

where W is a standard m_0-dimensional Brownian motion, $x \in \mathbb{R}^d$ and σ is the $(d \times m_0)$-matrix

$$\sigma = \begin{pmatrix} I_{m_0} \\ 0 \end{pmatrix}.$$

The solution of (2.1) is the Gaussian process

$$X_T^{t,x} = e^{(T-t)B} x - \int_t^T e^{(T-s)B} \sigma dW_s$$

whose transition density is

$$\Gamma_0(t, x; T, y)$$

$$= \frac{1}{\sqrt{(2\pi)^d \det \mathcal{C}(T-t)}} \exp\left\{ -\frac{1}{2} \langle \mathcal{C}(T-t)^{-1} \big(y - e^{(T-t)B} x\big), \big(y - e^{(T-t)B} x\big) \rangle \right\}$$

for $t < T$ and $x, y \in \mathbb{R}^d$. Here

$$\mathcal{C}(t) = \int_0^t \left(e^{sB}\sigma\right) \left(e^{sB}\sigma\right)^* ds$$

is the covariance matrix of $X_T^{t,x}$.

Remark 4
The assumption on the matrix B (i.e. to be of that specific block-form) is also equivalent to the fact that $\mathcal{C}(t)$ is positive definite for any $t > 0$.

Operator L_0 has some remarkable invariance properties that were first studied by *Lanconelli and Polidoro (1994)*. Denote by $\ell_{(\tau,\xi)}$, for $(\tau, \xi) \in \mathbb{R}^{d+1}$, the left-translations in $\mathbb{R}^{d+1}$ defined as

$$\ell_{(\tau,\xi)}(t, x) := (\tau, \xi) \circ (t, x) := \left(t + \tau, x + e^{tB}\xi\right).$$

Then, L_0 is invariant with respect to ℓ_ζ in the sense that

$$L_0\,(u \circ \ell_\zeta) = (L_0 u) \circ \ell_\zeta, \qquad \zeta \in \mathbb{R}^{d+1}.$$

Moreover, let

$$\mathcal{D}(r) := \mathrm{diag}(rI_{m_0}, r^3 I_{m_1}, \dots, r^{2\nu+1} I_{m_\nu}), \qquad r > 0,.$$

Then, L_0 is homogeneous with respect to the dilations in $\mathbb{R}^{d+1}$ defined as

$$\delta_r(t,x) := \left(r^2 t, \mathcal{D}(r)x\right), \qquad r > 0,$$

if and only if all the $*$-blocks of B are null. In that case, we have

$$L_0(u \circ \delta_r) = r^2 (L_0 u) \circ \delta_r, \qquad r > 0.$$

Since the Jacobian $J\mathcal{D}(r)$ equals r^Q, the natural number

$$Q = m_0 + 3m_1 + \cdots + (2\nu + 1)m_\nu.$$

is usually called the *homogeneous dimension* of $\mathbb{R}^d$ with respect to $(\mathcal{D}(r))_{r>0}$.

3 Inherited properties of L

It turns out that the invariance properties of the principal part L_0 are inherited by L in terms of *invariance within the class* $\mathcal{K}_{M,B}$. More explicitly, for the left-translations we have

Fact 1
Let $\zeta \in \mathbb{R}^{d+1}$ and $L \in \mathcal{K}_{M,B}$. If u is a solution of $Lu = 0$, then

$$v := u \circ \ell_\zeta \quad \textit{solves} \quad L^{(\zeta)} v = 0$$

where $L^{(\zeta)}$ is obtained from L by left-translating its coefficients, that is $L^{(\zeta)} = L \circ \ell_\zeta$. Moreover, operator $L^{(\zeta)}$ still belongs to $\mathcal{K}_{M,B}$.

As for dilations, we have to distinguish between *homogeneous Kolmogorov operators* (i.e. operators with null $*$-blocks in B) and general Kolmogorov operators.

Fact 2 (The homogeneous case)
Let $\lambda > 0$ and $L \in \mathcal{K}_{M,B}$ be a homogeneous Kolmogorov operator. If u is a solution of $Lu = 0$ then

$$v := u \circ \delta_\lambda \quad \textit{solves} \quad L^\lambda v = 0$$

where L^λ is obtained from L by dilating its coefficients, that is $L^\lambda = L \circ \delta_\lambda$. Moreover, operator L^λ still belongs to $\mathcal{K}_{M,B}$.

It turns out that the crucial step to achieve our estimate is to prove it for $t = 0$, $T = 1$ and $y = 0$, that is

$$\Gamma(0, x; 1, 0) \leq C \exp\left(-\frac{|x|^2}{C}\right), \qquad x \in \mathbb{R}^d,$$

with C *dependent only on* M *and* B. Then, the general estimate for $L \in \mathcal{K}_{M,B}$ follows from the invariance of the class $\mathcal{K}_{M,B}$ with respect to the left-translations ℓ and the intrinsic dilations δ.
This upper bound is consistent with the following Gaussian upper bound for Kolmogorov operators with *Hölder continuous coefficients*, proved in *Polidoro* and *Di Francesco and Pascucci* (see also *Konakov, Menozzi and Molchanov (2010)* and *Bally and Kohatsu-Higa (2015)*) by means of the classic parametrix method:

$$\Gamma(t, x; T, y) \leq C\Gamma_0(t, x; T, y), \qquad t < T,\ x \in \mathbb{R}^d, \tag{3.1}$$

where $C = C(M)$ and Γ_0 is the fundamental solution of L_0 with $\sigma = \begin{pmatrix} \sqrt{2M} I_{m_0} \\ 0 \end{pmatrix}$. Notice that for *homogeneous* Kolmogorov operators, the constant C in estimate (3.1) is independent of $T - t$.

In the case of *non-homogeneous* Kolmogorov operators, our main estimate is different and slightly less accurate than the Gaussian bound proved in *Pascucci and Polidoro (2003)*. Indeed, the lack of homogeneity makes the proof more involved since the scaling argument cannot be used anymore. We have the following result (see *Lanconelli and Polidoro (1994)*).

Fact 3 (Non-homogeneous case)
Let $\lambda > 0$ *and* $L \in \mathcal{K}_{M,B}$. *If* u *is a solution of* $Lu = 0$ *then*

$$v := u \circ \delta_\lambda \quad \textit{solves} \quad L^\lambda v = 0$$

where

$$L^\lambda u := \operatorname{div}(A^{(\lambda)} Du) + \langle B^{(\lambda)} x, Du\rangle + \partial_t u + \operatorname{div}(a^{(\lambda)} u) + c^{(\lambda)} u(t, x),$$

with

$$A^{(\lambda)}(t, x) = A\left(\delta_\lambda(t, x)\right), \quad a^{(\lambda)}(t, x) = \lambda a\left(\delta_\lambda(t, x)\right), \quad c^{(\lambda)}(t, x) = \lambda^2 c\left(\delta_\lambda(t, x)\right),$$

and $B^{(\lambda)} = \lambda^2 \mathcal{D}_\lambda B \mathcal{D}_{\frac{1}{\lambda}}$*, that is*

$$B^{(\lambda)} = \begin{pmatrix} \lambda^2 B_{1,1} & \lambda^4 B_{1,2} & \cdots & \lambda^{2\nu} B_{1,\nu} & \lambda^{2\nu+2} B_{1,\nu+1} \\ B_1 & \lambda^2 B_{2,2} & \cdots & \lambda^{2\nu-2} B_{2,\nu} & \lambda^{2\nu} B_{2,\nu+1} \\ 0 & B_2 & \cdots & \lambda^{2\nu-4} B_{3,\nu} & \lambda^{2\nu-2} B_{3,\nu+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_\nu & \lambda^2 B_{\nu+1,\nu+1} \end{pmatrix},$$

where $B_{i,j}$ *denotes the* $*$*-block in the* (i,j)*-th position of* B*.*

We will show that if $L \in \mathcal{K}_{M,B}$, then the fundamental solution Γ^λ of L^λ satisfies the main estimate uniformly with respect to $\lambda \in [0,1]$, that is with the constant C dependent only on M and B. Intuitively, this is due to the fact that, on the one hand, the dilations δ_λ do not affect the blocks $B_1, \dots, B_\nu$ in B (this guarantees the hypoellipticity of the operator, uniformly with respect to λ); on the other hand, the new $*$-blocks are bounded functions of $\lambda \in [0,1]$.

4 Moser's estimate

The first step in the proof of our main theorem consists in proving the local boundedness of non-negative weak solutions of $Lu = 0$. Let us first rewrite

$$\sum_{i,j=1}^{m_0} \partial_{x_i}(a_{ij}\partial_{x_j} u) + \sum_{i=1}^{m_0} \partial_{x_i}(a_i u) + cu + \sum_{i,j=1}^{d} b_{ij} x_j \partial_{x_i} u + \partial_t u$$

in the compact form

$$\mathrm{div}(ADu) + \mathrm{div}(au) + cu + Yu,$$

where $D = (\partial_{x_1}, \dots, \partial_{x_d})$ denotes the gradient in $\mathbb{R}^d$, $A := (a_{ij})_{1\le i,j\le d}$, $a := (a_i)_{1\le i\le d}$ with $a_{ij} = a_i \equiv 0$ for $i > m_0$ or $j > m_0$ and as before

$$Y = \langle Bx, D\rangle + \partial_t, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

We recall the definition of weak solution.

Definition 1
We say that u *is a* weak sub-solution *of* $Lu = 0$ *in a domain* Ω *of* $\mathbb{R}^{d+1}$ *if*

$$u, \partial_{x_1} u, \dots, \partial_{x_{m_0}} u, Yu \in L^2_{\mathrm{loc}}(\Omega)$$

and for any non-negative $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega -\langle ADu, D\varphi\rangle - \langle a, D\varphi\rangle u + \varphi c u + \varphi Y u \geq 0.$$

A function u is a weak super-solution *if $-u$ is a weak sub-solution. If u is a weak sub and super-solution, then we say that u is a* weak solution.

The following cylinders reflect the geometric properties of the operator L.

Definition 2
We denote

$$R_1 = \{(t, x) \in \mathbb{R} \times \mathbb{R}^d \mid |t| < 1,\, |x| < 1\};$$

moreover, for $z_0 \in \mathbb{R}^{d+1}$ and $r > 0$, we set

$$R_r(z_0) := z_0 \circ \delta_r(R_1) = \{z \in \mathbb{R}^{d+1} \mid z = z_0 \circ \delta_r(\zeta),\, \zeta \in R_1\}.$$

In the classical setting, Moser's approach combines Caccioppoli type estimates with the embedding Sobolev inequality. For Kolmogorov operators that are not uniformly parabolic, Caccioppoli estimates provide L^2_{loc}-bounds only for the first m_0 derivatives.

Theorem 4 (Caccioppoli type inequality)
Let $L \in \mathcal{K}_{M,B}$ and u be a non-negative weak sub-solution of $Lu = 0$ in $R_r(z_0)$, with $0 < \varrho < r \leq r_0$ such that $r - \varrho < 1$. If $u^q \in L^2(R_r(z_0))$ for some $q > \frac{1}{2}$, then $D_{m_0} u^q \in L^2(R_\rho(z_0))$ and there exists a constant $C = C(M, \|B\|)$ such that

$$\int_{R_\rho(z_0)} |D_{m_0} u^q|^2 \leq C \left(\frac{q}{2q-1}\right)^2 \frac{q}{(r-\rho)^2} \int_{R_r(z_0)} |u^q|^2.$$

If u is a non-negative weak super-solution, then the previous inequality holds for $q < \frac{1}{2}$.

Proof
Follows the same line of the proof for the uniformly parabolic case. □

Remark 5
Since the constant C from the inequality above depends only on M and $\|B\|$, the previous estimate holds also for for operator

$$L^\lambda u := \operatorname{div}(A^{(\lambda)} Du) + \langle B^{(\lambda)} x, Du\rangle + \partial_t u + \operatorname{div}(a^{(\lambda)} u) + c^{(\lambda)} u$$

uniformly with respect to $\lambda \in [0, 1]$.

The following result follows the original argument proposed in *Pascucci and Polidoro (2004)*, which consists in proving some ad hoc Sobolev type inequalities for local solutions to $Lu = 0$.

Theorem 5 (Sobolev type inequality)
Let $L \in \mathcal{K}_{M,B}$, $\lambda \in [0,1]$. If u is a non-negative weak sub-solution of $L^\lambda u = 0$ in $R_r(z_0)$, then $u \in L^{2\kappa}_{\mathrm{loc}}(R_r(z_0))$ with $\kappa = 1 + \frac{2}{Q}$ and we have

$$\|u\|_{L^{2\kappa}(R_\rho(z_0))} \leq \frac{C}{r-\rho}\left(\|u\|_{L^2(R_r(z_0))} + \|D_{m_0} u\|_{L^2(R_r(z_0))}\right),$$

for every $0 < \rho < r \leq r_0$, satisfying $r - \varrho < 1$, with C dependent only on M, B and r_0. The same statement holds for non-negative super-solutions.

Proof
Based on potential estimates obtained in *Pascucci and Polidoro (2004)* □

We are now ready to state the local boundedness for non negative weak solutions of $Lu = 0$.

Theorem 6
Let $L \in \mathcal{K}_{M,B}$, $\lambda \in [0,1]$ and u be a non-negative weak solution of $L^\lambda u = 0$ in a domain Ω. Let $z_0 \in \Omega$ and $0 < \varrho < r \leq r_0$ be such that $r - \varrho < 1$ and $\overline{R_r(z_0)} \subseteq \Omega$. Then, for every $p > 0$ there exists a positive constant $C = C(M, r_0, p)$ such that

$$\sup_{R_\varrho(z_0)} u^p \leq \frac{C}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} u^p.$$

The previous estimate also holds for every $p < 0$ such that $u^p \in L^1(R_r(z_0))$.

Remark 6
This result slightly extends the Moser's estimates obtained in Cinti, Pascucci and Polidoro (2008) *where the lower order terms were not included.*

Proof The argument is based on the Moser's iteration method. The inequality to be iterated, obtained combining the Caccioppoli and Sobolev type inequalities is

$$\|u^q\|_{L^{2\kappa}(R_\rho(z_0))} \leq \frac{C(M, r_0, q)\sqrt{|q|}}{(r-\varrho)^2}\|u^q\|_{L^2(R_r(z_0))},$$

where $0 < \rho < r \le r_0$ with $r - \rho < 1$, $q \neq \frac{1}{2}$ and u is a non-negative weak solution of $L^\lambda u = 0$. From the Caccioppoli-type inequality we see that $C(M, r_0, q)$, as a function of q, is bounded at infinity and diverges at $q = \frac{1}{2}$: this feature is in common with the equation studied in Cinti et al. (2008). However, the presence of the new factor $\sqrt{|q|}$ in the right hand side of that inequality requires additional care in the application of the Moser's iterative procedure. First of all, we fix a sequence of radii $\rho_n = \left(1 - \frac{1}{2^n}\right)\rho + \frac{1}{2^n} r$, a sequence of exponents $q_n = \frac{p}{2}\kappa^n$ and a safety distance, say δ, from $\frac{1}{2}$. The exponent p is chosen to guarantee that the distance of the resulting exponent q_n from $\frac{1}{2}$ is at least δ, for each $n \ge 1$. We then iterate the inequality above to obtain

$$\|u^{\frac{p}{2}}\|_{L^\infty(R_\rho(z_0))} \le f(r - \varrho)\|u^{\frac{p}{2}}\|_{L^2(R_r(z_0))}$$

where, for some $\widetilde{C} = \widetilde{C}(M, r_0, \delta)$,

$$f(r - \varrho) = \prod_{j=0}^{\infty} \left(\frac{\widetilde{C}\sqrt{|p|}\kappa^{\frac{j}{2}}}{(\varrho_j - \varrho_{j+1})^2} \right)^{\frac{1}{\kappa^j}} = \frac{C_1(M, r_0, p)}{(r - \varrho)^{\frac{Q+2}{2}}},$$

This proves the claim for p satisfying $|\frac{p}{2}k^n - \frac{1}{2}| \ge \delta$. The previous restriction is easily relaxed using the monotonicity of the L^p-means. $\square$

5 The upper bound

We are now going to prove a Gaussian upper bound for the fundamental solution Γ of $L \in \mathcal{K}_{M,B}$. The existence of Γ for Kolmogorov equations with Hölder continuous coefficients has been proved in *Weber (1951)*, *Il'in (1964)*, *Eidelman et al. (1998)* and, in greatest generality, in *Polidoro (1994)* and *Di Francesco and Pascucci (2005)* in the homogeneous and non-homogeneous cases, respectively.
We begin with an important implication of the Moser's estimate.

Theorem 7 (Nash upper bound)
Let Γ be a fundamental solution of $L \in \mathcal{K}_{M,B}$. Then, there exists a positive constant $C = C(M, T_0)$ such that

$$\Gamma(t, x; T, y) \le \frac{C}{(T - t)^{\frac{Q}{2}}}, \qquad 0 < T - t \le T_0,\ x, y \in \mathbb{R}^d.$$

Remark 7
We remark that the previous estimate can be interpreted as a Gaussian upper bound in a parabolic region. In fact, for fixed (T, y) *and* $\lambda > 0$, *let*

$$\mathcal{P}_\lambda(T, y) := \{(t, x) \mid |x - y| \le \lambda\sqrt{T - t}\}.$$

Then, the previous inequality obviously implies

$$\Gamma(t, x; T, y) \le \frac{C}{(T-t)^{\frac{Q}{2}}} e^{-\frac{|x-y|^2}{\lambda(T-t)}}, \qquad (t, x) \in \mathcal{P}_\lambda(T, y).$$

Our proof of a Gaussian upper bound for the fundamental solution is adapted to the approach of *Aronson (1967)*. The next theorem is a crucial step in this direction.

Theorem 8
Fix $y \in \mathbb{R}^d$, $\sigma > 0$ *and let* $u_0 \in L^2(\mathbb{R}^d)$ *be such that* $u_0(x) = 0$ *for* $|x - y| < \sigma$. *Let* $L \in \mathcal{K}_{M,B}$ *and suppose that* u *is a bounded solution in* $[\eta - \sigma^2, \eta[\, \times \mathbb{R}^d$ *with terminal value* $u(\eta, x) = u_0(x)$. *Then, there exist positive constants* k *and* C *such that for any* τ *which satisfies* $\eta - \frac{1 \wedge \sigma^2}{k} \le \tau \le \eta$ *we have*

$$|u((0, e^{-\eta B} y) \circ (\tau, 0))| \le C(\eta - \tau)^{-\frac{Q}{4}} \exp\left(-\frac{\sigma^2}{C(\eta - \tau)}\right) \|u_0\|_{L^2(\mathbb{R}^d)}.$$

The constants k *and* C *depend only on* M.

Proof
Consider the case $y = 0$. We fix s such that $0 \le \eta - s \le 1 \wedge \sigma^2$ and we define

$$h(t, x) = -\frac{|x|^2}{2(\eta - s) - k(\eta - t)} + \alpha(\eta - t), \qquad \eta - \frac{\eta - s}{k} \le t \le \eta, \; x \in \mathbb{R}^d,$$

with α and k being positive constants to be fixed later on. Moreover, for $R \ge 2$, we consider a function $\gamma_R \in C_0^\infty(\mathbb{R}^d, [0, 1])$ such that $\gamma_R(x) \equiv 1$ for $|x| \le R - 1$, $\gamma_R(x) \equiv 0$ for $|x| \ge R$ with $|D\gamma_R|$ bounded by a constant independent of R. Then, we multiply both sides of the equation by $\gamma_R^2 e^{2h} u$ and we integrate over $[\tau, \eta] \times \mathbb{R}^d$, with $\eta - \frac{\eta - s}{k} \le \tau \le \eta$, to get

$$\int_{\mathbb{R}^d} \gamma_R^2 e^{2h} u^2|_{t=\tau}\, dx - 2 \iint_{[\tau,\eta]\times\mathbb{R}^d} e^{2h} u^2 \left(3\langle A D_{m_0} h, D_{m_0} h\rangle - Yh - 2\langle a, D_{m_0} h\rangle + \Lambda\right) dxdt$$

$$\le \int_{\mathbb{R}^d} \gamma_R^2 e^{2h} u^2|_{t=\eta}\, dx + 2 \iint_{[\tau,\eta]\times\mathbb{R}^d} e^{2h} u^2 \left(3\mu \left|D_{m_0}\gamma_R\right|^2 + \left|Y\gamma_R^2\right| - 2\langle a, D_{m_0}\gamma_R\rangle \gamma_R\right) dxdt,$$

where Λ is a positive constant depending on M.
Next, we let R go to infinity: since u is bounded by assumption and

$$e^{2h(t,x)} \leq e^{-\frac{|x|^2}{\eta-s}+2\alpha(\eta-s)},$$

the last integral tends to zero and we get

$$\int_{\mathbb{R}^d} e^{2h}u^2|_{t=\tau}\,dx - 2\iint_{[\tau,\eta]\times\mathbb{R}^d} e^{2h}u^2\left(3\langle AD_{m_0}h, D_{m_0}h\rangle - Yh - 2\langle a, D_{m_0}h\rangle + \Lambda\right)dxdt \leq \int_{\mathbb{R}^d} e^{2h}u^2|_{t=\eta}\,dx.$$

Then, by a suitable choice of k and α, only dependent on M, B, we have

$$3\langle AD_{m_0}h, D_{m_0}h\rangle - Yh - 2\langle a, D_{m_0}h\rangle + \Lambda \leq 0, \qquad \eta - \frac{\eta-s}{k} \leq t \leq \eta,\ x \in \mathbb{R}^d.$$

Hence, we derive the inequalities

$$\max_{t\in\,]\eta-\frac{\eta-s}{k},\eta[} \int_{\left|D\left(\frac{2\sqrt{k}}{\sqrt{\eta-s}}\right)x\right|\leq 1} e^{2h(t,x)}u^2(t,x)dx \leq \max_{t\in\,]\eta-\frac{\eta-s}{k},\eta[} \int_{\mathbb{R}^d} e^{2h(t,x)}u^2(t,x)dx \leq \int_{|x|\geq\sigma} e^{2h(\eta,x)}u_0^2(x)dx.$$

Now we notice that, by definition, for every $t \in\,]\eta - \frac{\eta-s}{k}, \eta]$ we have

$$\begin{aligned} 2h(t,x) &\geq -\frac{2|x|^2}{\eta-s} \\ &= -\frac{2\left|\mathcal{D}(\delta)\mathcal{D}(\delta^{-1})x\right|^2}{\eta-s} \\ &\geq -\frac{2\left\|\mathcal{D}(\delta)\right\|^2}{\eta-s} \\ &\geq -\frac{2\delta^2}{\eta-s} = -\frac{1}{2k}. \end{aligned}$$

On the other hand, if $|x| \geq \sigma$, we have

$$-2h(\eta,x) = \frac{2|x|^2}{2(\eta-s)} \geq \frac{\sigma^2}{\eta-s}.$$

Using the previous estimates, we get

$$\max_{t\in]\eta-\frac{\eta-s}{k},\eta[} \int_{\left|D\left(\frac{2\sqrt{k}}{\sqrt{\eta-s}}\right)x\right|\le 1} u^2(t,x)dx \le e^{\frac{1}{2k}} \exp\left(-\frac{\sigma^2}{\eta-s}\right) \|u_0\|^2_{L^2(\mathbb{R}^d)}.$$

Finally, we rely on Moser's estimate in order to get the desired inequality. We let $\tau = \eta - \frac{\eta-s}{k}$ and we observe that $\tau \in [\eta - \frac{1}{k}, \eta]$ and $\eta - s = k(\eta - \tau)$: thus we have

$$\begin{aligned}
|u(\tau,0)|^2 &\le \sup_{R^+_{\frac{\sqrt{\eta-s}}{4\sqrt{k}}}(\tau,0)} |u|^2 \\
&\le \frac{C}{(\eta-s)^{\frac{Q+2}{2}}} \iint_{R^+_{\frac{\sqrt{\eta-s}}{2\sqrt{k}}}(\tau,0)} u^2(t,x)dxdt \\
&= \frac{C}{(\eta-s)^{\frac{Q+2}{2}}} \int_{\tau}^{\tau+\frac{\eta-s}{4k}} \int_{\left|D\left(\frac{2\sqrt{k}}{\sqrt{\eta-s}}\right)x\right|\le 1} u^2(t,x)dxdt \\
&\le \frac{C}{(\eta-s)^{\frac{Q}{2}}} \exp\left(-\frac{\sigma^2}{C(\eta-s)}\right) \|u_0\|^2_{L^2(\mathbb{R}^d)} \\
&= \frac{C}{k^{\frac{Q}{2}}(\eta-\tau)^{\frac{Q}{2}}} \exp\left(-\frac{\sigma^2}{Ck(\eta-\tau)}\right) \|u_0\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where the constant $C = C(M,k)$. This yields the claim in the case $y = 0$. □

The following corollary is a simple consequence of the previous theorem.

Corollary 1
There exists two positive constants k and C, that depend only on M, such that for every $\sigma > 0$ and $\eta \in \mathbb{R}$, we have

$$\int_{\left|\xi - e^{(\eta-t)B}x\right|\ge\sigma} \Gamma^2(t,x;\eta,\xi)d\xi \le \frac{Ce^{-\frac{\sigma^2}{C(\eta-t)}}}{(\eta-t)^{\frac{Q}{2}}}, \qquad (t,x) \in \left[\eta - \frac{1\wedge\sigma^2}{k}, \eta\right[\times \mathbb{R}^d,$$

and

$$\int_{\left|x - e^{(t-\eta)B}\xi\right|\ge\sigma} \Gamma^2(t,x;\eta,\xi)dx \le \frac{Ce^{-\frac{\sigma^2}{C(\eta-t)}}}{(\eta-t)^{\frac{Q}{2}}}, \qquad (t,x) \in \left[\eta - \frac{1\wedge\sigma^2}{k}, \eta\right[\times \mathbb{R}^d.$$

Proof

First of all we observe that

$$\int\limits_{\left|\xi-e^{(\eta-t)B}x\right|\geq\sigma} \Gamma^2(t,x;\eta,\xi)d\xi = \int\limits_{|\xi-y|\geq\sigma} \Gamma^2\left(t,e^{(t-\eta)B}y;\eta,\xi\right)d\xi$$

$$= \int\limits_{|\xi-y|\geq\sigma} \Gamma^2\left((0,e^{-\eta B}y)\circ(t,0);\eta,\xi\right)d\xi.$$

Now, the function

$$u(s,w) := \int\limits_{|\xi-y|\geq\sigma} \Gamma(s,w;\eta,\xi)\Gamma((0,e^{-\eta B}y)\circ(t,0);\eta,\xi)d\xi,$$

is a non-negative solution to our Kolmogorov equation for $s<\eta$, with terminal condition

$$u(\eta,w) = \begin{cases} 0 & \text{if } |w-y|<\sigma, \\ \Gamma((0,e^{-\eta B}y)\circ(t,0);\eta,w) & \text{if } |w-y|\geq\sigma. \end{cases}$$

Setting $(s,w)=(0,e^{-\eta B}y)\circ(t,0)$, we infer

$$\int\limits_{|\xi-y|\geq\sigma} \Gamma^2\left((0,e^{-\eta B}y)\circ(t,0);\eta,\xi\right)d\xi = u((0,e^{-\eta B}y)\circ(t,0))$$

$$\leq \frac{Ce^{-\frac{\sigma^2}{C(\eta-t)}}}{(\eta-t)^{\frac{Q}{4}}}\|\Gamma\left((0,e^{-\eta B}y)\circ(t,0),\eta,\cdot\right)\|_{L^2(\mathbb{R}^d)}.$$

Then, the thesis follows directly from the corollary of Nash inequality. □

We are now in position to prove our main result.

Theorem 9

Let $L\in\mathcal{K}_{M,B}$ and $T_0>0$. If Γ is a fundamental solution of the operator L, then there exists a positive constant C, only dependent on M, B and T_0, such that

$$\Gamma(t,x;T,y) \leq \frac{C}{(T-t)^{\frac{Q}{2}}}\exp\left(-\frac{1}{C}\left|\mathcal{D}\left((T-t)^{-\frac{1}{2}}\right)\left(x-e^{-(T-t)B}y\right)\right|^2\right),$$

for $0<T-t\leq T_0$ and $x,y\in\mathbb{R}^d$.

Proof

Step 1. We first prove the thesis for $y=0$ and $T-t=\frac{1}{k}$, with k as in the previous theorem (*Aronson*). We fix $x \in \mathbb{R}^d$ and set

$$\sigma(x) = \frac{|x|}{2\|e^{\frac{T-t}{2}B}\|}.$$

If $\sigma(x) \le 1$, that is $|x| \le 2\|e^{\frac{T-t}{2}B}\|$, then the thesis is a direct consequence of Nash inequality and the fact that, by assumption, $T-t=\frac{1}{k}$ is fixed with k dependent only upon M.
On the other hand, if $\sigma(x) \ge 1$, by the Chapman-Kolmogorov identity and putting $\eta = T - \frac{T-t}{2}$, we have

$$\Gamma(t,x;T,0) = \int_{\mathbb{R}^d} \Gamma(t,x;\eta,\xi)\Gamma(\eta,\xi;T,0)d\xi = J_1 + J_2,$$

where

$$J_1 := \int_{\left|\xi - e^{\frac{T-t}{2}B}x\right| \ge \sigma(x)} \Gamma(t,x;\eta,\xi)\Gamma(\eta,\xi;T,0)d\xi,$$

$$J_2 := \int_{\left|\xi - e^{\frac{T-t}{2}B}x\right| < \sigma(x)} \Gamma(t,x;\eta,\xi)\Gamma(\eta,\xi;T,0)d\xi.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(J_1)^2 &\le \int_{\left|\xi - e^{\frac{T-t}{2}B}x\right| \ge \sigma(x)} \Gamma^2(t,x;\eta,\xi)d\xi \int_{\left|\xi - e^{\frac{T-t}{2}B}x\right| \ge \sigma(x)} \Gamma^2(\eta,\xi;T,0)d\xi \\
&\le \frac{Ce^{-\frac{\sigma^2(x)}{C(T-t)}}}{(T-t)^Q} \\
&= Ck^Q \exp\left(-\frac{k|x|^2}{4C\|e^{\frac{1}{2k}B}\|^2}\right).
\end{aligned}$$

In order to estimate J_2, we first note that if $\left|\xi - e^{\frac{T-t}{2}B}x\right| < \sigma(x)$ then, recalling also the definition of $\sigma(x)$, we have

$$|\xi| \ge \left|e^{-\frac{T-t}{2}}x\right| - \left|\xi - e^{-\frac{T-t}{2}}x\right| \ge \frac{|x|}{\|e^{\frac{T-t}{2}B}\|} - \sigma(x) = \sigma(x).$$

Thus, from the previous estimate and using again the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(J_2)^2 &\leq \int\limits_{|\xi|\geq\sigma(x)} \Gamma^2(\eta,\xi;T,0)d\xi \int\limits_{|\xi|\geq\sigma(x)} \Gamma^2(t,x;\eta,\xi)d\xi \\
&\leq \frac{Ce^{-\frac{\sigma^2(x)}{C(T-t)}}}{(T-t)^{\frac{Q}{2}}} \int\limits_{\mathbb{R}^d} \Gamma^2(t,x;\eta,\xi)d\xi \\
&\leq \frac{C}{(T-t)^Q} e^{-\frac{\sigma^2(x)}{C(T-t)}} \\
&= Ck^Q \exp\left(-\frac{k|x|^2}{4C\|e^{\frac{1}{2k}B}\|^2}\right).
\end{aligned}$$

This completes the proof of the case $\sigma(x) \geq 1$. In conclusion, we have proved the desired estimate for $T-t=\frac{1}{k}$, that is

$$\Gamma(t,x;T,0) \leq Ce^{-\frac{|x|^2}{C}}, \qquad T-t=\frac{1}{k},\ x\in\mathbb{R}^d,$$

with the constant C only dependent on M and B. Actually, the same estimate holds also for the fundamental solution Γ^λ of L^λ, *with C independent of $\lambda\in[0,1]$:* in fact, all the results derive from the Moser's estimate which is uniform in $\lambda\in[0,1]$.

Step 2. We use a scaling argument to generalize the last estimate to the case $0<T-t\leq\frac{1}{k}$; precisely, we prove that

$$\Gamma(t,x;T,0) \leq \frac{C}{(T-t)^{\frac{Q}{2}}} e^{-\frac{|x|^2}{C(T-t)}}, \qquad 0<T-t\leq\frac{1}{k},\ x\in\mathbb{R}^d.$$

For $\lambda\in[0,1]$, we set

$$\Gamma^\lambda(t,x;T,0) = \lambda^Q \Gamma(\delta_\lambda(t,x);\delta_\lambda(T,0))$$

and observe that, since the Jacobian $J\mathcal{D}(\lambda)$ equals λ^Q, we have that Γ^λ is a fundamental solution of the operator $L^{(\lambda)}$.
Now, fix t such that $0<T-t\leq\frac{1}{k}$ and set $\lambda=k(T-t)$. Then we have

$$\Gamma(t,x;T,0) = \lambda^{-\frac{Q}{2}} \Gamma^{(\sqrt{\lambda})}\left(\frac{t}{\lambda}, \mathcal{D}\left(\frac{1}{\sqrt{\lambda}}\right)x; \frac{T}{\lambda}, 0\right) \leq$$

(by (5))

$$\leq C\lambda^{-\frac{Q}{2}} e^{-\frac{1}{C}\left|\mathcal{D}\left(\frac{1}{\sqrt{\lambda}}\right)x\right|^2}$$

which proves the claim.

Step 3. We now remove the condition $y = 0$. Let $z = (0, e^{-TB}y)$ and $\Gamma^{(z)}$ be the fundamental solution of the operator $L^{(z)} := L \circ \ell_z$. Since $L^{(z)} \in \mathcal{K}_{M,B}$, we have that $\Gamma^{(z)}$ satisfies the last estimate and hence we obtain

$$\begin{aligned}
\Gamma(t,x;T,y) &= \Gamma^{(z)}(z^{-1} \circ (t,x); T, 0) \\
&= \Gamma^{(z)}(t, x - e^{-(T-t)B}y; T, 0) \\
&\leq \frac{C}{(T-t)^{\frac{Q}{2}}} \exp\left(-\frac{1}{C}\left|\mathcal{D}\left(\frac{1}{\sqrt{T-t}}\right)\left(x - e^{-(T-t)B}y\right)\right|^2\right),
\end{aligned}$$

for $0 < T - t \leq \frac{1}{k}$ and $x, y \in \mathbb{R}^d$.

Step 4. In the last step we relax the restriction on the length of the time interval. We first suppose that $0 < T - t \leq \frac{2}{k}$ and set $\tau = \frac{T-t}{2}$. By the Chapman-Kolmogorov identity we have

$$\begin{aligned}
\Gamma(t,x;T,y) &= \int_{\mathbb{R}^d} \Gamma(t,x;t+\tau,\xi)\Gamma(t+\tau,\xi;T,y)d\xi \\
&\leq \frac{C}{\tau^Q} \int_{\mathbb{R}^d} e^{-\frac{1}{C}\left|\mathcal{D}\left(\frac{1}{\sqrt{\tau}}\right)\left(x - e^{-\tau B}\xi\right)\right|^2} e^{-\frac{1}{C}\left|\mathcal{D}\left(\frac{1}{\sqrt{\tau}}\right)\left(\xi - e^{-\tau B}y\right)\right|^2} d\xi \\
&\leq \frac{C}{\tau^Q} \int_{\mathbb{R}^d} e^{-\frac{1}{C}\left|\mathcal{D}\left(\frac{1}{\sqrt{\tau}}\right)\left(x - e^{-\tau B}\xi\right)\right|^2} e^{-\frac{1}{C}\left|\mathcal{D}\left(\frac{1}{\sqrt{\tau}}\right)\left(e^{-\tau B}\xi - e^{-(T-t)B}y\right)\right|^2} d\xi \\
&\leq \frac{C}{(T-t)^{\frac{Q}{2}}} e^{-\frac{1}{C}\left|\mathcal{D}\left(\frac{1}{\sqrt{T-t}}\right)\left(x - e^{-(T-t)B}y\right)\right|^2}.
\end{aligned}$$

Iterating this procedure we can extend the estimate to any bounded time interval and this concludes the proof. □