About the interplay between Partial differential equations and probability.

The first contact of probability and p.d.e passes through the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u \qquad x \in R^d,\ t \geq 0 \qquad (1)$$

which is satisfied by the transition function of the d-dimensional Brownian motion.

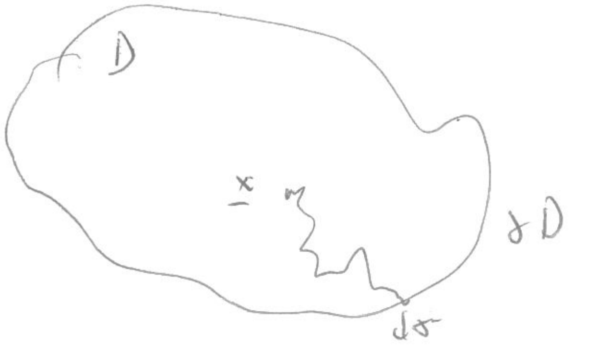
A fruitful ~~contact~~ relationship between these areas of mathematics is established by the so-called theory of probabilistic potential which analyses two main features of Brownian motion, that is the escape hitting distribution and the mean exit time from a set.

Let us consider a set D with boundary ∂D, suitably regular.

The following probability

$$P_x\{ B(T_{\partial D}) \in d\sigma \mid B(0) = \underline{x} \} = u(\underline{x}) \qquad (2)$$

where $\underline{x} \in R^d$, $T_{\partial D} = \inf\{ t : B(t) \in \partial D \}$, B is a d- Brownian motion without drift



is a solution of the Laplace equation

$$\Delta u = 0 \qquad x \in D$$

In particular the Dirichlet problem

$$\begin{cases} \Delta u = 0 & x \in D \\ u = h & x \in \partial D \end{cases}$$

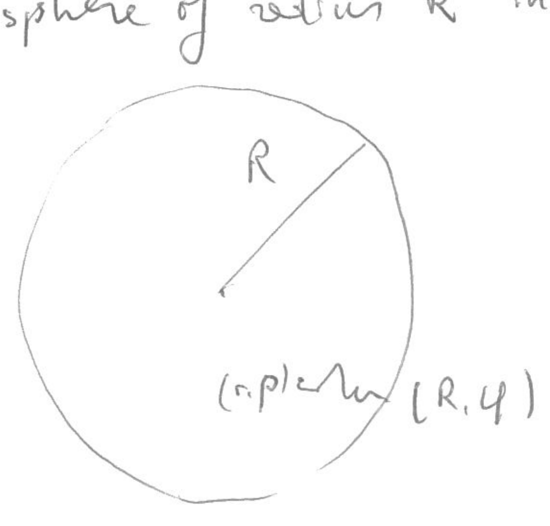
admits the solution

$$u(\underline{x}) = E\{ h(B(T_{\partial D})) \mid B(0) = \underline{x} \}$$

For example, in the plane, we are able to show that

$$P\{ \underline{B}(T_{\partial S}) \in d\sigma \mid \underline{B}(0) = (\rho, r) \} = \frac{R^2 - \rho^2}{2\pi \left[R^2 + \rho^2 - 2\rho R \cos(\theta - \varphi) \right]}$$

where (ρ, θ) are the polar coordinates of the starting point (x_1, x_2) and (R, φ) the coordinates of the exit point on the circumference of radius R.
Of course, it is possible to give the distribution (Poisson kernel) of the exit point of a Brownian motion from a sphere of radius R in the Euclidean space R^d.



For the mean exist time

$$E\left[T_{\partial D} \mid \underline{B}(0) = \underline{x}\right] = v(\underline{x})$$

we know that the function v solves the Poisson equation

$$\frac{1}{2}\Delta v \doteq -1$$

And this fundamental information permits us to obtain in the case of a sphere that is

$$E\left[T_{\partial D} \mid B(0) = \cancel{x}\right] = \frac{R^2 - \|x\|^2}{d}$$

It is also possible to show that for the Poincaré half-plane the hitting probability of the hyperbolic Brownian motion $(X(t), Y(t))$

$$\begin{cases} dX = Y\, dB_1 \\ dY = X\, dB_2 \end{cases}$$

that is

$$P_r\{B(T_{\partial D}) \in d\sigma \mid B(0) = \underline{x}\}$$

satisfies the hyperbolic Laplacian

$$\frac{Y^2}{2}\Delta u = 0$$

It is possible to show that if D is a hyperbolic sphere the hitting probability

$$P_r\{B^h(T_{\partial S}) \in d(\eta, \alpha) \mid B^h(0) = (\bar{\eta}, \bar{\alpha})\} =$$

$$\frac{1}{2\pi}\,\frac{\operatorname{ch}\bar{\eta} - \operatorname{ch}\eta}{\operatorname{ch}\eta\operatorname{ch}\bar{\eta} - 1 - \operatorname{sh}\eta\operatorname{sh}\bar{\eta}\cos(\alpha - \bar{\alpha})}$$

Another interesting interaction between stochastic processes and p.d.e. concerns fractional calculus and stable process.

A stable process $S_\alpha(\sigma, \beta, \mu; t)$ is a Lévy process with characteristic function

$$E e^{i\theta S_\alpha(\sigma,\beta,\mu;t)} = e^{-|\theta|^\alpha \sigma^\alpha t\left(1 - i\beta \operatorname{sign}\theta \operatorname{tg}\frac{\pi\alpha}{2}\right) + i\theta\mu t}$$

where $\sigma > 0$, $-1 \leq \beta \leq 1$, $0 < \alpha \leq 2$, $\alpha \neq 1$, $\mu \in R$.

For $\alpha = 1$ we have that

$$E e^{i\theta S_1(\sigma,\beta,\mu;t)} = e^{-|\theta|\sigma t\left(1 + i\beta \frac{2}{\pi} \lg|\theta|\right) + i\mu\theta t}$$

For $0 < \alpha < 1$, $\beta = 1$ we have the stable subordinators taking values in $(0, \infty)$. For $\mu = 0$, $\beta = 0$, we have the symmetric stable processes with ch. function

$$E e^{i\beta S_\alpha(t)} = e^{-|\theta|^\alpha t}$$

It is possible to show that the distribution p_α of the stable process $S_\alpha(t)$ satisfies the space-fractional equation

$$\frac{\partial p_\alpha}{\partial t} = \frac{\partial^\alpha p_\alpha}{\partial |x|^\alpha}$$

with initial condition $p_\alpha(x, 0) = f(x)$ (and boundary conditions $\lim_{|x| \to \infty} p_\alpha(x,t)$ $\lim_{|x| \to \infty} \frac{dp_\alpha}{dx}(x, t) = 0$

The operator $\frac{\partial^\alpha}{\partial |x|^\alpha}$ is called the Riesz derivative and is defined as

$$\frac{\partial^\alpha f(x)}{\partial|x|^\alpha} = -\frac{1}{2\cos\frac{\alpha\pi}{2}\,\Gamma(1-\alpha)}\frac{d^m}{dx^m}\int_{-\infty}^{+\infty}\frac{f(z)}{|x-z|^\alpha}\,dz$$

where $m-1<\alpha<m$. In the present case we have $0<\alpha<2$ so that for $0<\alpha<1$, $m=1$, and for $1<\alpha<2$, $m=2$.
The Fourier transform of the Riesz derivative reads

$$\int_{-\infty}^{+\infty} e^{i\theta x}\frac{\partial^\alpha f}{\partial|x|^\alpha}(x)\,dx = -|\theta|^\alpha\int_{-\infty}^{+\infty} e^{i\xi x} f(x)\,dx$$

And this clearly shows that the Fourier transform of equation

$$\frac{\partial p_\alpha}{\partial t} = \frac{\partial^\alpha p_\alpha}{\partial|x|^\alpha}$$

is

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} e^{i\theta x}p_\alpha(x,t)\,dx = -|\xi|^\alpha\int_{-\infty}^{\infty} e^{i\theta x}p_\alpha(x,t)\,dx$$

is

$$E e^{i\theta S_\alpha(t)} = \int_{-\infty}^{+\infty} e^{i\theta x}p_\alpha(x,t)\,dx = e^{-|\theta|^\alpha t}$$

For the asymmetric stable processes we need to consider the Riesz-Feller space fractional derivative, $|\theta| \leq \min(\alpha, 2-\alpha)$

$$\left({}_xD^\alpha_\theta f\right)(x) = \frac{\Gamma(1-\alpha)}{\pi}\left\{\sin\frac{\pi(\alpha+\theta)}{2}\int_0^\infty\frac{f(x+\xi)-f(x)}{\xi^{1+\alpha}}\,d\xi + \sin\frac{\pi(\alpha-\theta)}{2}\int_0^\infty\frac{f(x-\xi)-f(x)}{\xi^{1+\alpha}}\,d\xi\right\}$$

which for $\theta=0$ becomes the Riesz derivative, The Fourier transform of the R.F. derivative reads

$$\mathcal{F}\left({}_xD_\theta^\alpha f\right)(\gamma) = -|\gamma|^\alpha e^{i\left(\frac{\theta\pi}{2}\right)\operatorname{sgn}\gamma}$$

and thus the equation

$$\frac{\partial}{\partial t} p_\alpha(x,t) = \left({}_xD_\theta^\alpha f\right) p_\alpha(x,t)$$

has Fourier transform

$$\frac{\partial E e^{i\gamma S_\alpha(t)}}{\partial t} = -|\gamma|^\alpha e^{i\frac{\theta\pi}{2}\operatorname{sgn}\gamma}$$

whose solution is the ch. function of a stable process with $\mu = 0$, that is

$$E e^{i\gamma S_\alpha(t)} = e^{-|\gamma|^\alpha t\left(\cos\frac{\pi\theta}{2} + i\operatorname{sgn}\gamma \sin\frac{\pi\theta}{2}\right)}$$

$$= e^{-|\gamma|^\alpha t \cos\frac{\pi\theta}{2}\left(1 + i\operatorname{sgn}\gamma \tan\frac{\pi\theta}{2}\right)}$$

$$= e^{-|\gamma|^\alpha t \sigma^\alpha\left(1 - i\operatorname{sgn}\gamma\,\beta \tan\frac{\pi\alpha}{2}\right)}$$

$$\sigma = \cos\frac{\pi\theta}{2}, \quad \beta = -\frac{\tan\frac{\pi\theta}{2}}{\tan\frac{\pi\alpha}{2}}$$

For the stable subordinator $H_\nu(t) = S_\nu(\sigma, 1, 0; t)$ we know that the distribution $h_\nu(x, t)$ is the solution of the Cauchy problem

$$\frac{\partial h_\nu}{\partial t} = - \frac{\partial^\nu h_\nu}{\partial x^\nu} \qquad x > 0$$

$$t > 0$$

$$h_\nu(0, t) = 0 \qquad 0 < \nu < 1$$

$$h_\nu(x, 0) = 0$$

and $\frac{\partial^\nu}{\partial x^\nu}$ is the Riemann-Liouville derivative.

For the inverse process $L^\nu(t)$ of the stable subordinator $H^\nu(t)$ defined as

$$L^\nu(t) : \inf(s \geq 0 ; H^\nu(s) \geq t\}$$

we have that its law $l_\nu(t, x)$ satisfies the time-fractional equation

$$\frac{\partial^\nu l_\nu}{\partial t^\nu} = - \frac{\partial l_\nu}{\partial x}$$

with $l_\nu(x, 0) = f(x)$

By the way consider that we are able to give the explicit distributions $h_\nu(t, x)$ and $l_\nu(t, x)$ in terms of Wright functions as

$$h_\nu(t, x) = \Pr\{H^\nu(t) \in dx\}/dx =$$

$$= \frac{\nu t}{x^{\nu+1}} W_{-\nu, 1-\nu}\left(-\frac{t}{x^\nu}\right) \qquad \begin{matrix} x > 0 \\ t > 0 \end{matrix}$$

For example, for $\nu = \frac{1}{2}$, $\nu = \frac{1}{3}$ we have that

$$h_{\frac{1}{2}}(x,t) = \frac{t}{\sqrt{4\pi x^3}}\, e^{-\frac{t^2}{4x}} \qquad x \geq 0$$

$$h_{\frac{1}{3}}(x,t) = \frac{t}{x\sqrt[3]{3x}} \cdot Ai\left(\frac{t}{\sqrt[3]{3x}}\right) \qquad x \geq 0$$

For the inverse process $L_\nu(t)$ the distribution $\ell_\nu(x,t)$ becomes

$$\ell_\nu(x,t) = \Pr\{L_\nu(t) \in dx\} = \frac{1}{t^\nu} W_{-\nu,1-\nu}\left(-\frac{x}{t^\nu}\right) \qquad x > 0$$

and

$$\ell_{\frac{1}{2}}(x,t) = \frac{1}{\sqrt{\pi t}}\, e^{-\frac{x^2}{4t}} \qquad x > 0$$

$$\ell_{\frac{1}{3}}(x,t) = \frac{3}{\sqrt[3]{3t}}\, Ai\left(\frac{x}{\sqrt[3]{3t}}\right) \qquad x > 0$$

Hyperbolic equations.

A field where the connection between stochastic processes and p.d.e. is less known is that of hyperbolic equations and random motions of finite velocity.
The simplest case is that of the asymmetric telegraph process with rightward velocity c_1 and leftward velocity $-c_2$ (and exponential intertimes with rates λ_1, λ_2).
The position at time t of the moving particle, say $T = T(t)$, $t > 0$ is located in the interval

$$[-c_2 t, +c_1 t]$$

the endpoints being reached when no Poisson event occurs. Within the interval $[-c_2 t, c_1 t]$, the distribution

$$p(x,t)\,dx = \Pr\{T(t) \in dx\}$$

satisfies the hyperbolic equation

$$\frac{\partial^2 p}{\partial t^2} = c_1 c_2 \frac{\partial^2 p}{\partial x^2} + (c_2 - c_1)\frac{\partial^2 p}{\partial x \partial t} - (\lambda_1 + \lambda_2)\frac{\partial p}{\partial t}$$
$$+ \frac{1}{2}\Big[(c_2 - c_1)(\lambda_1 + \lambda_2) - (\lambda_2 - \lambda_1)(c_1 + c_2)\Big]\frac{\partial p}{\partial x}$$

For $c_1 = c_2 = c$, $\lambda_1 = \lambda_2 = \lambda$ the above equation reduces to

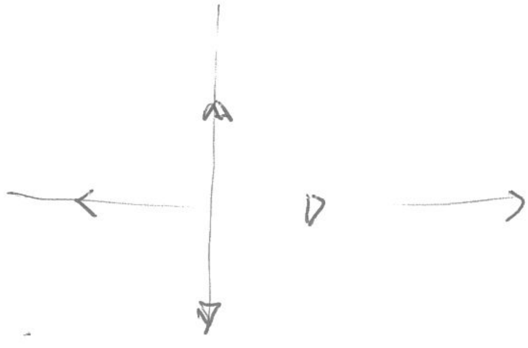
$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} - 2\lambda \frac{\partial p}{\partial t}$$

It is possible to extract from the above equation the explicit form of the distribution $p(x,t)$ by means of the following procedure. By applying the Lorentz transformation of the special relativity one can eliminate the drift of the process. This corresponds to passing from the first equation to the second one written in a different coordinate frame (x', t'). This permits to obtain the governing equation

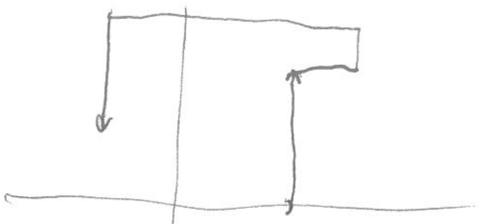
$$\frac{\partial^2 p}{\partial t'^2} = \frac{4(c_1 + c_2)^2 \lambda_1^2 \lambda_2^2}{(\lambda_1 + \lambda_2)^4}\frac{\partial^2 p}{\partial x'^2} - \frac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}\frac{\partial p}{\partial t'}$$

Since the explicit distribution of $T(t)$ in the symmetric case is well-known we can arrive at the distribution in the non-symmetric $T(t)$ by returning to the original coordinates (x,t).

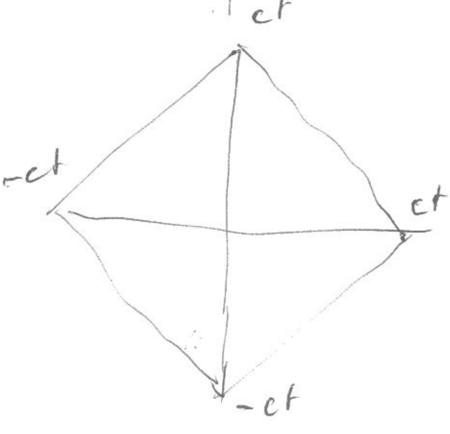
Random motions at finite velocity with a finite number of directions produces also higher-order hyperbolic equations and the order of these equations coincides with the number of possible directions of motion.
This is an important drawback because it easily leads to complicated, cumbersome equations.
For a simple example we take the motion in the plane with four orthogonal directions chosen initially with equal probability $\frac{1}{4}$.



The instant of occurrence of a Poisson event the moving particle changes direction and takes the orthogonal direction in either of the two possible ways.



The sample paths of motion are thus broken lines of random length. At an arbitrary time t the particle is located within a square of vertices $(0,ct)$, $(ct,0)$, $(-ct,0)$, $(0,-ct)$. The distribution



$$p(x,y,t)\,dx\,dy = \Pr\{X(t)\in dx,\ Y(t)\in dy\}$$

inside the square satisfies the fourth-order equation

$$\left(\frac{\partial}{\partial t}+\lambda\right)^2\left(\frac{\partial^2}{\partial t^2}+2\lambda-c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right)p+c^4\frac{\partial^4 p}{\partial x^2\partial y^2}=0$$

The explicit distribution of p inside the square coincides with the distribution of

$$\begin{cases} X(t) = U(t) + V(t) \\ Y(t) = U(t) - V(t) \end{cases}$$

where U and V are independent one-dimensional Telegraph equations with velocities $\pm \frac{c}{2}$ and rate $\frac{\lambda}{2}$.

We note that if the number of directions is infinite (at least in some cases) we do not have the unpleasant effect of facing equations of increasing order.

In the planar random motion with uniformly distributed directions taken at Poisson times the law of $(X(t), Y(t))$

$$p(x,y,t)\,dx\,dy = \Pr\{X(t) \in dx,\ Y(t) \in dy\}$$

satisfies the planar telegraph equation

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = c^2 \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right)$$

which is known as the equations of damped vibrations of membranes.

Non-linear equations in probability.

Perhaps the best known equation ~~non~~-linear equation considered by probabilists is that called the Burgers equation written in the form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

which can be reduced to the heat equation by means of the so-called Cole-Hopf transformation which is transformed into the heat equation by means of the Cole-Hopf transformation

$$u = -2\nu \frac{1}{\phi} \frac{\partial \phi}{\partial x}$$

into the heat equation

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2}$$

From my point of view a more important non-linear equation which has an impact on probability is the Barenblatt equation

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad m > 1$$

which admits the so called Kompaneets-Zeldovich solution in the form

$$u(x,t) = Ct^{\delta}\left(1 - B\frac{\|x\|^2}{t^{\eta}}\right)_+^{\gamma}$$

where parameters depend on m and the dimension d of space. For example $\gamma = \frac{1}{m-1}$, $\eta = \frac{2}{2+m(d-1)}$ $\delta = -\frac{d}{2+m(d-1)}$

The solution looks like the fundamental solution of the EPD equation

$$\left(\frac{\partial^2}{\partial t^2} + \frac{d+2\gamma-1}{t}\frac{\partial}{\partial t}\right)p(\underline{x},t) = c^2\sum_{j=1}^{d}\frac{\partial^2 p}{\partial x_j^2}$$

which reads

$$p(\underline{x},t) = \frac{\Gamma(\gamma + \frac{d}{2})}{\pi^{d/2}\,\Gamma(\gamma)}\,\frac{1}{(ct)^{d-2+2\gamma}}\,\left(c^2t^2 - \|x\|^2\right)^{\gamma-1}$$

Proof of the linearization of the Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad (1.1)$$

with the Hopf-Cole Transformation

$$u = -2\nu \frac{\frac{\partial \varphi}{\partial x}}{\varphi}$$

We rewrite equation (1.1) as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 - \nu\frac{\partial u}{\partial x}\right) = 0$$

and then apply the transformation

$$\frac{\partial}{\partial t}\left(-2\nu\frac{\frac{\partial \varphi}{\partial x}}{\varphi}\right) + \frac{\partial}{\partial x}\left(\cancel{\frac{1}{2}\cdot 2^2\nu^2\frac{\left(\frac{\partial \varphi}{\partial x}\right)^2}{\varphi^2}} - \nu(-2\nu)\left\{\frac{\frac{\partial^2\varphi}{\partial x^2}\cdot\varphi - \cancel{\left(\frac{\partial \varphi}{\partial x}\right)^2}}{\varphi^2}\right\}\right) = 0$$

which becomes

$$\frac{\partial}{\partial t}\frac{1}{\varphi}\frac{\partial \varphi}{\partial x} - \nu\frac{\partial}{\partial x}\left(\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{1}{\varphi}\right) = 0$$

and since

$$\frac{\partial}{\partial t}\left(\frac{1}{\varphi}\frac{\partial \varphi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\varphi}\frac{\partial \varphi}{\partial t}\right) \quad \text{we have}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\varphi}\frac{\partial \varphi}{\partial t}\right) - \nu\frac{\partial}{\partial x}\left(\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{1}{\varphi}\right) = 0$$

and integrating w.r.t. x we get

$$\frac{\partial \varphi}{\partial t} = \nu\frac{\partial^2\varphi}{\partial x^2}$$

In the case of branching and diffusion of Brownian particles, the function

$$u(x,t) = E_x\left\{\prod_{j=1}^{N(t)} f(B_j(t))\right\}$$

where $B_j(t)$, $j \geq 1$ are independent Brownian motions, $N(t)$ is a linear birth process, f a suitable bounded function, the function $u = u(x,t)$ satisfies the non-linear K-P-P equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \lambda u^2 - \lambda u$$

where λ is the rate of the birth process. The term u^2 is related to the fact that each particle can produce one offspring. Under the assumption of an arbitrary number of births the non-linear term has an exponent equal to $n+1$ if n is the number of generated offsprings.

A last remark

The joint distribution $p(x,y,t)$ of a Brownian motion B and and its integral $\int_0^t B(s)ds$ satisfies the linear equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2} + y\frac{\partial p}{\partial x}$$