UNIMORE
UNIVERSITÀ DEGLI STUDI DI MODENA E REGGIO EMILIA

Kolmogorov-Fokker-Planck Equations: theoretical issues and applications

April 10 - 11, 2017
Modena

Efficient Method for Barrier Option Evaluation

C. Guardasoni, S. Sanfelici
University of Parma, Italy

Outline

Semi-**A**nalytical method for the pricing of **B**arrier **O**ptions, under general dynamics.

In practice…
the extension of Boundary Element Method, introduced in the Engineering field in the 1970s, to **barrier option pricing**
here in a user-friendly way

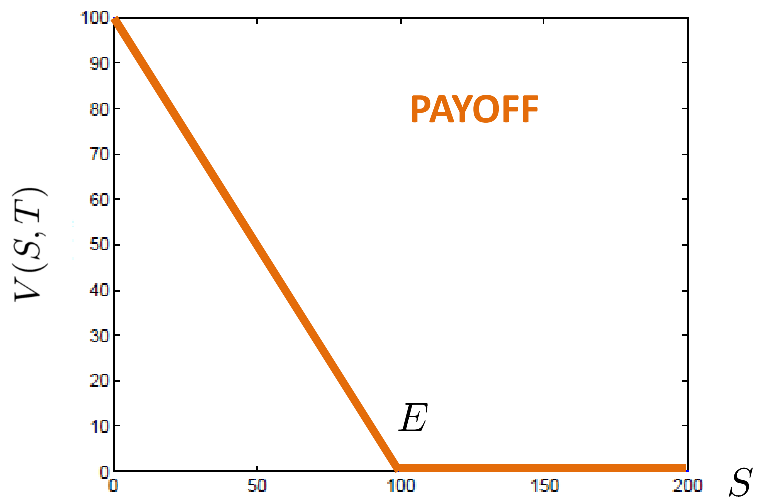
Requirement:
Knowledge of the **fundamental solution** (transition probability density function)
related to the differential model problem associated to the vanilla option
at least in an approximated form

- Black-Scholes model problem
- Foundations
- Numerical examples
- Straightforward application to hedging
- Extension to Heston model

The financial model problem: European barrier option pricing

A **European option** $V(S,t)$ is a contract
which gives the buyer
the right to sell (**put** option) or to buy (**call** option) an underlying asset S
at a specified strike price E
on a specified date (expiry) T

At **maturity** T, for **Put Option** with exercise (strike) price E:
if $S \leq E$, the holder can buy the underlying asset at S and exercise the right to sell it at E, thus the option's value is $E - S$.
On the contrary, if $S > E$, why sell something at a price E that is lower than its market price? Thus, if $S > E$, the option is not exercised and the holder receives zero.



The financial model problem: European barrier option pricing

a **knock-out barrier option** is an option whose price extinguishes when the underlying asset breaches a pre-set **barrier level**

For clarity, I will illustrate here only the case of a

European put up-and-out option
whose price extinguishes when the underlying asset breaches a pre-set **upper barrier level**

but the method is analogously applicable also to call option and other combinations of barriers too.

The mathematical model problem: European vanilla option

Under the simple **Black-Scholes paradigm**, still very common in use with **time dependent** parameters $\sigma(t), r(t), d(t)$

European vanilla option differential model problem

- $V(x,t)$ option price $\quad x = \log(S) \in (-\infty, +\infty),\ t \in [0,T)$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \left(r - \frac{\sigma^2}{2} - d\right)\frac{\partial V}{\partial x} - rV = 0$$

- with **final condition** (payoff)

$$V(x,T) = \max(E - e^x, 0) \qquad x \in (-\infty, +\infty)$$

- with **boundary conditions** on the asset

$$\lim_{x\to-\infty} V(x,t) = Ee^{-\int_t^T r\,d\tau} \qquad \lim_{x\to+\infty} V(x,t) = 0 \qquad t \in [0,T)$$

$S =$ underlying asset value
$r =$ interest rate
$d =$ dividend yield
$\sigma =$ volatility
$T =$ expiry
$E =$ exercise price

For this problem the **analytical solution** is known

$$N[q] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{q} e^{-y^2/2}dy$$ normal cumulative distribution; $$q = -\frac{\log(e^x/E) + \int_t^T (r - \delta + \sigma^2/2)d\tau}{\left(\int_t^T \sigma^2 d\tau\right)^{1/2}}$$

$$V(x,t) = Ee^{-\int_t^T r\,d\tau} N\left[q + \left(\int_t^T \sigma^2\,d\tau\right)^{1/2}\right] - e^{x-\int_t^T d\,d\tau} N[q]$$

[F. Black and M. Scholes, 1973]

The mathematical model problem: European vanilla option

following the PDE theory,
the analytical solution can be written as the discounted expected value of the final payoff

$$V(x,t) = e^{-\int_t^T r d\tau} \int_{-\infty}^{+\infty} V(y,T) G(y,T;x,t) dy$$

where $V(y,T)$ is the payoff and

$G(y,\tau;x,t)$ is the **fundamental solution** of the forward PDE

$$\begin{cases} -\dfrac{\partial G}{\partial \tau} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 G}{\partial y^2} - (r - \dfrac{\sigma^2}{2} - \delta)\dfrac{\partial G}{\partial y} - rG = 0 & \tau > t \\ G(y,t,x,t) = \delta(x,y) & \end{cases}$$

[A. Friedman, 1964-1975-1976]

The mathematical model problem: European put up-and-out option

$$\boxed{S \in [0, S_u]} \quad \text{and} \quad t \in [0, T]$$

Performing these classical **changes of variables**

$$V(S,t) = u(S,t)e^{-\int_t^T r(t')dt'} \qquad S = e^x \qquad \tau = T - t$$

and defining $r(t) = r(T-\tau) =: \bar{r}(\tau), \quad \sigma(t) = \sigma(T-\tau) =: \bar{\sigma}(\tau), \quad \text{and} \quad d(t) = d(T-\tau) =: \bar{d}(\tau)$

European put up-and-out option differential model problem

- $\dfrac{\partial u}{\partial \tau} - \dfrac{\bar{\sigma}^2}{2}\dfrac{\partial^2 u}{\partial x^2} - (\bar{r} - \dfrac{\bar{\sigma}^2}{2} - \bar{d})\dfrac{\partial u}{\partial x} = 0 \qquad x \in \Omega = (-\infty, U), \ \tau \in (0, T]$
- with **initial condition**

  $u(x, 0) = \max(E - e^x, 0) =: u_0(x) \qquad x \in \Omega$
- with **boundary conditions** on the asset

  $\lim_{x \to -\infty} u(x, \tau) = E \qquad u(U, \tau) = 0 \qquad \tau \in [0, T]$

$S =$ underlying asset value
$\bar{r} =$ interest rate
$\bar{d} =$ dividend yield
$\bar{\sigma} =$ volatility
$T =$ expiry
$E =$ exercise price
$U = \log(\text{upper barrier})$

Is there a closed form solution?

Numerical methods

- **Monte Carlo methods**: very simple and flexible, but also very slow to converge
- **Binomial/trinomial lattices**: relatively easy to implement, but not particularly efficient
- **Finite difference schemes**: easy to implement. However, standard high-order implementations fail to achieve true high-order accuracy, due to the non-smoothness of the options' payoffs
- **Finite element methods**: very accurate and fast and capable of handling discontinuous solutions; However, they are quite difficult to implement, especially if a high-degree polynomial basis is employed and have some troubles particularly in unbounded domains (as Finite Difference methods)

SABO Foundations

Semi-**A**nalytical method for the pricing of **B**arrier **O**ptions, under general dynamics.
based on **Boundary Element Method**

Foundations:

- *Analytical* Integral Representation of PDE solution
- Boundary Integral Equation
- Numerical Resolution of the Boundary Integral Equation by Collocation Method
- *Numerical approximation* of the option price

Integral Representation Formula of the PDE Solution

following **PDE theory…**

PDE $$\frac{\partial u}{\partial \tau} - \frac{\bar{\sigma}^2}{2}\frac{\partial^2 u}{\partial x^2} - (\bar{r} - \frac{\bar{\sigma}^2}{2} - \bar{d})\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tau}(x,\tau) - \mathcal{L}[u](x,\tau) = 0 \qquad x \in \Omega = (-\infty, U),\ \tau \in (0,T]$$

the related **transition probability density** (Green fundamental solution)

$$G(y,s,x,\tau) = \frac{1}{\sqrt{2\pi \int_s^\tau \bar{\sigma}^2(v)dv}} \exp\left\{ -\frac{\left[y - x - \int_s^\tau \left(\bar{r} - \frac{\bar{\sigma}^2}{2} - \bar{d}\right)(v)dv\right]^2}{2\int_s^\tau \bar{\sigma}^2(v)dv} \right\}, \qquad \tau > s$$

for each $(x,\tau) \in \mathbb{R} \times [0,T)$, $G(y,s,x,\tau)$ solves

$$\begin{cases} -\frac{\partial G}{\partial s}(y,s;x,\tau) - \mathcal{L}^*[G](y,s;x,\tau) = 0 & y \in \mathbb{R},\ s < \tau \\ G(y,\tau;x,\tau) = \delta(x,y) & y \in \mathbb{R} \end{cases}$$

Multiplying the **PDE** by G, integrating by parts (**Green's Theorem**)
and using **initial/boundary conditions**

$$u(x,\tau) = \int_\Omega u(y,0)G(y,0,x,\tau)dy$$

for each $x \in \Omega = (-\infty, U),\ \tau \in (0,T]$

Integral Representation Formula of the PDE Solution

following **PDE theory…**

PDE $$\frac{\partial u}{\partial \tau} - \frac{\bar\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - (\bar r - \frac{\bar\sigma^2}{2} - \bar d)\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tau}(x,\tau) - \mathcal{L}[u](x,\tau) = 0 \qquad x \in \Omega = (-\infty, U),\ \tau \in (0,T]$$

the related transition probability density (**Green fundamental solution**)

$$G(y,s,x,\tau) = \frac{1}{\sqrt{2\pi \int_s^\tau \bar\sigma^2(v)dv}} \exp\left\{ -\frac{\left[y - x - \int_s^\tau (\bar r - \frac{\bar\sigma^2}{2} - \bar d)(v)dv\right]^2}{2\int_s^\tau \bar\sigma^2(v)dv} \right\}, \quad \tau > s$$

for each $(x,\tau) \in \mathbb{R} \times [0,T)$, $G(y,s,x,\tau)$ solves

$$\begin{cases} -\frac{\partial G}{\partial s}(y,s;x,\tau) - \mathcal{L}^*[G](y,s;x,\tau) = 0 & y \in \mathbb{R},\ s < \tau \\ G(y,\tau;x,\tau) = \delta(x,y) & y \in \mathbb{R} \end{cases}$$

Multiplying the **PDE** by G, integrating by parts (**Green's Theorem**)
and using **initial/boundary conditions**

$$u(x,\tau) = \int_\Omega u(y,0)G(y,0,x,\tau)dy + \int_0^\tau \int_{\partial\Omega} \frac{\bar\sigma^2(s)}{2}\frac{\partial u}{\partial y}(y,s)G(y,s,x,\tau)dyds$$

for each $x \in \Omega = (-\infty, U),\ \tau \in (0,T]$

Integral Representation Formula of the PDE Solution

following **PDE theory…**

PDE

$$\frac{\partial u}{\partial \tau} - \frac{\bar\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - (\bar r - \frac{\bar\sigma^2}{2} - \bar d)\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tau}(x,\tau) - \mathcal{L}[u](x,\tau) = 0 \qquad x \in \Omega = (-\infty, U),\ \tau \in (0,T]$$

the related transition probability density (**Green fundamental solution**)

$$G(y,s,x,\tau) = \frac{1}{\sqrt{2\pi \int_s^\tau \bar\sigma^2(v)dv}} \exp\left\{ -\frac{\left[y - x - \int_s^\tau \left(\bar r - \frac{\bar\sigma^2}{2} - \bar d\right)(v)dv\right]^2}{2\int_s^\tau \bar\sigma^2(v)dv} \right\}, \qquad \tau > s$$

for each $(x,\tau) \in \mathbb{R} \times [0,T)$, $\quad G(y,s,x,\tau)$ solves

$$\begin{cases} -\frac{\partial G}{\partial s}(y,s;x,\tau) - \mathcal{L}^*[G](y,s;x,\tau) = 0 & y \in \mathbb{R},\ s < \tau \\ G(y,\tau;x,\tau) = \delta(x,y) & y \in \mathbb{R} \end{cases}$$

Multiplying the **PDE** by G, integrating by parts (**Green's Theorem**)
and using **initial/boundary conditions**

RF

$$\begin{aligned} u(x,\tau) &= \int_\Omega u(y,0)G(y,0,x,\tau)dy + \int_0^\tau \int_{\partial\Omega} \frac{\bar\sigma^2(s)}{2}\frac{\partial u}{\partial y}(y,s)G(y,s,x,\tau)dyds \\ &= \int_{-\infty}^U u_0(y)G(y,0,x,\tau)dy + \int_0^\tau \frac{\bar\sigma^2(s)}{2}\frac{\partial u}{\partial y}(U,s)G(U,s,x,\tau)ds \end{aligned}$$

for each $\quad x \in \Omega = (-\infty, U),\ \tau \in (0,T]$

Boundary Integral Equation

analytical **INTEGRAL REPRESENTATION FORMULA**

RF

$$u(x,\tau) = \int_{-\infty}^{U} u_0(y) G(y,0,x,\tau)\,dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2} \frac{\partial u}{\partial y}(U,s) G(U,s,x,\tau)\,ds$$

for each $x \in \Omega = (-\infty, U),\ \tau \in (0,T]$

unknown density

Boundary Integral Equation

analytical **INTEGRAL REPRESENTATION FORMULA**

RF

$$u(x,\tau) = \int_{-\infty}^{U} u_0(y) G(y,0,x,\tau)\,dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2} \frac{\partial u}{\partial y}(U,s) G(U,s,x,\tau)\,ds$$

for each $x \in \Omega = (-\infty, U),\ \tau \in (0,T]$

unknown density

but on the boundary, letting $x \to U$, **BOUNDARY INTEGRAL EQUATION**

BIE

$$0 = u(U,\tau) := \int_{-\infty}^{U} u_0(y) G(y,0;U,\tau)\,dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2} \frac{\partial u}{\partial y}(U,s) G(U,s;U,\tau)\,ds$$

for each $\tau \in (0,T]$

solve the equation… *numerically*

Boundary Integral Equation

analytical **INTEGRAL REPRESENTATION FORMULA**

RF

$$u(x,\tau) = \int_{-\infty}^{U} u_0(y)G(y,0,x,\tau)dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2}\frac{\partial u}{\partial y}(U,s)G(U,s,x,\tau)ds$$

for each $x \in \Omega = (-\infty, U),\ \tau \in (0,T]$

unknown density

but on the boundary, letting $x \to U,$ **BOUNDARY INTEGRAL EQUATION**

BIE

$$0 = u(U,\tau) := \int_{-\infty}^{U} u_0(y)G(y,0;U,\tau)dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2}\frac{\partial u}{\partial y}(U,s)G(U,s;U,\tau)ds$$

for each $\tau \in (0,T]$

solve the equation... *numerically*

Boundary Integral Equation

analytical **INTEGRAL REPRESENTATION FORMULA**

RF

$$u(x,\tau) = \int_{-\infty}^{U} u_0(y)G(y,0,x,\tau)dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2}\frac{\partial u}{\partial y}(U,s)G(U,s,x,\tau)ds$$

for each $x \in \Omega = (-\infty, U)$, $\tau \in (0,T]$

unknown density

but on the boundary, letting $x \to U$, **BOUNDARY INTEGRAL EQUATION**

BIE

$$0 = u(U,\tau) := \int_{-\infty}^{U} u_0(y)G(y,0;U,\tau)dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2}\frac{\partial u}{\partial y}(U,s)G(U,s;U,\tau)ds$$

for each $\tau \in (0,T]$

solve the equation… *numerically*

Note!: when $U \to +\infty$
the method reduces to the evaluation of the payoff expected value

Numerical Resolution of the Boundary Integral Equation

by **COLLOCATION METHOD**:

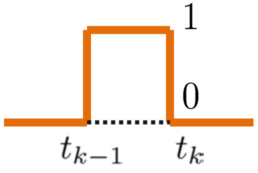
• uniform decomposition of the time interval $[0,T]$ with time step

$$\Delta t = T/N_{\Delta t} : \qquad t_k = k\Delta t \quad k = 0, \ldots, N_{\Delta t}$$

• approximation of the BIE unknown

$$\frac{\partial u}{\partial y}(U,s) \approx \phi(s) := \sum_{k=1}^{N_{\Delta t}} \alpha_k \varphi_k(s)$$

with $\varphi_k(s) := H[s - t_{k-1}] - H[s - t_k]$ for $k = 1, \ldots, N_{\Delta t}$



• evaluation of BIE at the **collocation nodes**: $\bar{t}_j = \dfrac{t_j + t_{j-1}}{2} \qquad j = 1, \ldots, N_{\Delta t}$

BIE

$$0 = u(U,\tau) := \int_{-\infty}^{U} u_0(y) G(y,0;U,\tau)\,dy + \int_0^{\tau} \frac{\partial u}{\partial y}(U,s) \frac{\overline{\sigma}^2(s)}{2} G(U,s;U,\tau)\,ds$$

Numerical Resolution of the Boundary Integral Equation

by **COLLOCATION METHOD**:

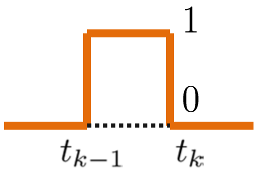
- uniform decomposition of the time interval $[0,T]$ with time step

$$\Delta t = T/N_{\Delta t}: \qquad t_k = k\Delta t \quad k = 0, \ldots, N_{\Delta t}$$

- approximation of the BIE unknown

$$\frac{\partial u}{\partial y}(U,s) \approx \phi(s) := \sum_{k=1}^{N_{\Delta t}} \alpha_k \varphi_k(s)$$

with $\varphi_k(s) := H[s - t_{k-1}] - H[s - t_k]$ for $k = 1, \ldots, N_{\Delta t}$



- evaluation of BIE at the **collocation nodes**: $\bar{t}_j = \dfrac{t_j + t_{j-1}}{2} \qquad j = 1, \ldots, N_{\Delta t}$

$$0 = u(U,\bar{t}_j) = \int_{-\infty}^{U} u_0(y) G(y,0;U,\bar{t}_j)dy + \int_0^{\bar{t}_j} \sum_{k=0}^{N_{\Delta t}-1} \alpha_k \varphi_k(s) \frac{\overline{\sigma}^2(s)}{2} G(U,s;U,\bar{t}_j)ds$$

Numerical Resolution of the Boundary Integral Equation

by **COLLOCATION METHOD**:

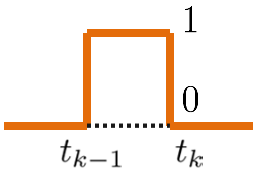
- uniform decomposition of the time interval $[0,T]$ with time step

$$\Delta t = T/N_{\Delta t}: \qquad t_k = k\Delta t \quad k = 0,\dots,N_{\Delta t}$$

- approximation of the BIE unknown

$$\frac{\partial u}{\partial y}(U,s) \approx \phi(s) := \sum_{k=1}^{N_{\Delta t}} \alpha_k \varphi_k(s)$$

with $\varphi_k(s) := H[s - t_{k-1}] - H[s - t_k]$ for $k = 1,\dots,N_{\Delta t}$



- evaluation of BIE at the **collocation nodes**: $\bar{t}_j = \dfrac{t_j + t_{j-1}}{2} \qquad j = 1,\dots,N_{\Delta t}$

$$\sum_{k=1}^{N_{\Delta t}} \alpha_k \underbrace{\int_0^{\bar{t}_j} \varphi_k(s) \frac{\overline{\sigma}^2(s)}{2} G(U,s;U,\bar{t}_j)\,ds}_{\mathcal{A}_{jk}} = \underbrace{-\int_{-\infty}^{U} u_0(y) G(y,0;U,\bar{t}_j)\,dy}_{\mathcal{F}_j}$$

Numerical Resolution of the Boundary Integral Equation

$$\mathcal{A}\alpha = \mathcal{F}$$

$$\mathcal{A} = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ A_{N_{\Delta t}1} & A_{N_{\Delta t}2} & \cdots & A_{N_{\Delta t}N_{\Delta t}-1} & A_{N_{\Delta t}N_{\Delta t}} \end{pmatrix}$$

as the Green's function is defined for $\tau > s$

$$\mathcal{A}_{jk} = \int_0^{\bar{t}_j} \varphi_k(s) \frac{\overline{\sigma}^2(s)}{2} G(U, s; U, \bar{t}_j) ds = \int_{t_{k-1}}^{\min(t_k, \bar{t}_j)} \frac{\overline{\sigma}^2(s)}{2\sqrt{2\pi \int_s^{\bar{t}_j} \overline{\sigma}^2(v) dv}} \exp\left\{ -\frac{\left[\int_s^{\bar{t}_j} \left(\overline{r} - \frac{\overline{\sigma}^2}{2} - \overline{d}\right)(v) dv\right]^2}{2\int_s^{\bar{t}_j} \overline{\sigma}^2(v) dv} \right\} ds$$

$j, k = 1, \ldots, N_{\Delta t}$, $j \geq k$

… here in a user-friendly way:

numerical integration is simply performed by adaptive quadrature functions of **Matlab**:

- **`quad`**
- and **`quadgk`** for weak singularity in matrix diagonal entries

Numerical Resolution of the Boundary Integral Equation

$$\mathcal{A}\alpha = \mathcal{F}$$

$$\mathcal{A} = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ A_{N_{\Delta t}1} & A_{N_{\Delta t}2} & \cdots & A_{N_{\Delta t}N_{\Delta t}-1} & A_{N_{\Delta t}N_{\Delta t}} \end{pmatrix}$$

as the Green's function is defined for $\tau > s$

$$\mathcal{A}_{jk} = \int_0^{\bar{t}_j} \varphi_k(s) \frac{\overline{\sigma}^2(s)}{2} G(U, s; U, \bar{t}_j) ds = \int_{t_{k-1}}^{\min(t_k, \bar{t}_j)} \frac{\overline{\sigma}^2(s)}{2\sqrt{2\pi \int_s^{\bar{t}_j} \overline{\sigma}^2(v) dv}} \exp\left\{ -\frac{\left[\int_s^{\bar{t}_j} \left(\overline{r} - \frac{\overline{\sigma}^2}{2} - \overline{d}\right)(v) dv\right]^2}{2\int_s^{\bar{t}_j} \overline{\sigma}^2(v) dv} \right\} ds$$

$j, k = 1, \ldots, N_{\Delta t}\,,\; j \geq k$

N.B.: if $\sigma,\ r,\ \delta$ are constant then

$$\mathcal{A} = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ A_2 & A_1 & 0 & \cdots & 0 \\ A_3 & A_2 & A_1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ A_{N_{\Delta t}} & A_{N_{\Delta t-1}} & \cdots & A_{N_2} & A_{N_1} \end{pmatrix}$$

Numerical Resolution of the Boundary Integral Equation

$$\mathcal{A}\alpha = \mathcal{F}$$

$$\mathcal{A} = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ A_{N_{\Delta t}1} & A_{N_{\Delta t}2} & \cdots & A_{N_{\Delta t}N_{\Delta t}-1} & A_{N_{\Delta t}N_{\Delta t}} \end{pmatrix}$$

as the Green's function is defined for $\tau > s$

$$\mathcal{A}_{jk} = \int_0^{\bar{t}_j} \varphi_k(s) \frac{\overline{\sigma}^2(s)}{2} G(U, s; U, \bar{t}_j) ds = \int_{t_{k-1}}^{\min(t_k, \bar{t}_j)} \frac{\overline{\sigma}^2(s)}{2\sqrt{2\pi \int_s^{\bar{t}_j} \overline{\sigma}^2(v) dv}} \exp\left\{ -\frac{\left[\int_s^{\bar{t}_j} \left(\overline{r} - \frac{\overline{\sigma}^2}{2} - \overline{d} \right)(v) dv \right]^2}{2 \int_s^{\bar{t}_j} \overline{\sigma}^2(v) dv} \right\} ds$$

$j, k = 1, \ldots, N_{\Delta t}\,,\; j \geq k$

$$\mathcal{A}\alpha = \mathcal{F} \longrightarrow \alpha$$

Numerical Approximation of the option price

analytical **INTEGRAL REPRESENTATION FORMULA**

RF

$$u(x,\tau) = \int_{-\infty}^{U} u_0(y) G(y,0,x,\tau)\,dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2} \frac{\partial u}{\partial y}(U,s) G(U,s,x,\tau)\,ds$$

for each $x \in \Omega = (-\infty, U)$, $\tau \in (0,T]$

unknown density

Numerical Approximation of the option price

approximation of **INTEGRAL REPRESENTATION FORMULA**

RF

$$u(x,\tau) \approx \int_{-\infty}^{U} u_0(y) G(y,0,x,\tau)\,dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2} \sum_{k=1}^{N_{\Delta t}} \alpha_k \varphi_k(s) G(U,s,x,\tau)\,ds$$

for each $x \in \Omega = (-\infty, U),\ \tau \in (0,T]$

approximation

$$u(x,\tau) \approx \int_{-\infty}^{\min(U,\log(E))} \frac{(E-e^y)}{\sqrt{2\pi \int_0^\tau \overline{\sigma}^2(v)dv}} \exp\left\{ -\frac{\left[y - x - \int_0^\tau \left(\overline{r} - \frac{\overline{\sigma}^2}{2} - \overline{d}\right)(v)dv\right]^2}{2\int_0^\tau \overline{\sigma}^2(v)dv} \right\} dy +$$

$$+ \sum_{k=1}^{\mathrm{ceil}[\frac{t}{\Delta t}]} \alpha_k \int_{t_{k-1}}^{\min(t_k,\tau)} \frac{\overline{\sigma}^2(s)}{2} \frac{1}{\sqrt{2\pi \int_s^\tau \overline{\sigma}^2(v)dv}} \exp\left\{ -\frac{\left[U - x - \int_s^\tau \left(\overline{r} - \frac{\overline{\sigma}^2}{2} - \overline{d}\right)(v)dv\right]^2}{2\int_s^\tau \overline{\sigma}^2(v)dv} \right\} ds$$

$\forall S \in (0, S_u)\,,\ \forall t \in [t_0, T)$ $\qquad V(S,t) = u(\log(S), T-t)e^{-\int_t^T r(t')dt'}$

Hedging

This numerical strategy is very useful and efficient also for hedging that needs computing ***Greeks*** because it is sufficient to evaluate the derivative of the RF

Δ - Hedging **without computing the primary unknown** V

- $$\Delta := \frac{\partial V}{\partial S} = \frac{1}{S}\frac{\partial u}{\partial x}(log(S), T-t)e^{-\int_t^T r(t')dt'}$$

$$\frac{\partial u}{\partial x}(x,\tau) := \int_{-\infty}^{U} u_0(y)\frac{\partial G}{\partial x}(y,0;x,\tau)dy + \int_0^{\tau}\frac{\overline{\sigma}^2(s)}{2}\frac{\partial G}{\partial x}(U,s;x,\tau)\frac{\partial u}{\partial y}(U,s)ds$$

$$\frac{\partial G}{\partial x}(y,s,x,\tau) = G(y,s,x,\tau)\frac{y-x-\int_s^{\tau}\left(\overline{r}-\frac{\overline{\sigma}^2}{2}-\overline{d}\right)(v)dv}{\int_s^{\tau}\overline{\sigma}^2(v)dv}$$

BOUNDARY INTEGRAL EQUATION

BIE

$$0 = u(U,\tau) := \int_{-\infty}^{U} u_0(y)G(y,0;U,\tau)dy + \int_0^{\tau}\frac{\overline{\sigma}^2(s)}{2}\frac{\partial u}{\partial y}(U,s)G(U,s;U,\tau)ds$$

for each $\tau \in (0,T]$

solve the equation… *numerically*

Numerical Example: test with constant parameters

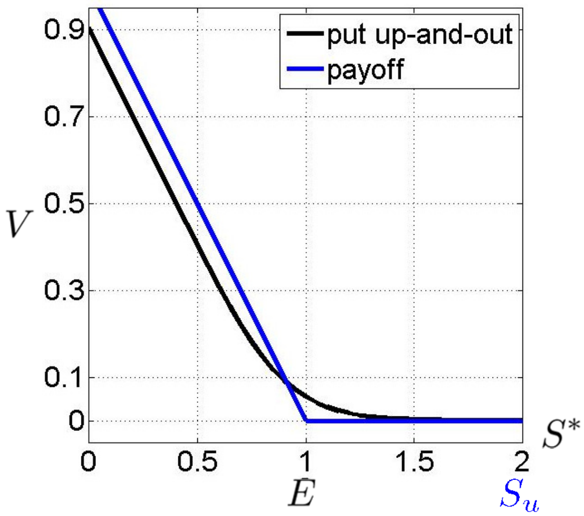
[L.V. Ballestra – G. Pacelli, 2014]

$\sigma = 0.25$ constant volatility
$E = 1$ exercise price
$r = 0.1$ interest rate
$\delta = 0$ dividend yield
$T = 1$ maturity

$e^{x^*} = S^* = [0 : 0.05 : 2]$
current underlying asset values

$S_u = 2$ upper barrier $> E$

$V(S^*, 0)$



closed-form solution **[J.C. Hull, 2011]**

$$
\begin{cases}
V(S,t) = Ee^{-r(T-t)}\mathcal{N}[y_1 + (1-2\lambda\sigma)\sqrt{T-t})] \\
\qquad -Se^{-\delta(T-t)}\mathcal{N}[y_1 - 2\lambda\sigma\sqrt{T-t}] \\
\qquad +Se^{-\delta(T-t)}(S_u/S)^{2\lambda}\mathcal{N}[-y_1] \\
\qquad -Ee^{-r(T-t)}(S_u/S)^{2\lambda-2}\mathcal{N}[-y_1 + \sigma\sqrt{(T-t)}] & \text{if } S_u \le E\,, \\
V(S,t) = P + Se^{-\delta(T-t)}(S_u/S)^{2\lambda}\mathcal{N}[-y] \\
\qquad -Ee^{-r(T-t)}(S_u/S)^{2\lambda-2}\mathcal{N}[-y + \sigma\sqrt{(T-t)}] & \text{if } S_u \ge E\,,
\end{cases}
$$

$$
\lambda = \frac{r - \delta + \sigma^2/2}{\sigma^2}\,; \quad y_1 = \frac{\log(S_u/S)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}\,; \quad y = \frac{\log\left(S_u^2/(SE)\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}\,;
$$

$P(S,t)$ is the value of the European put option without barriers

Numerical Example: test with constant parameters

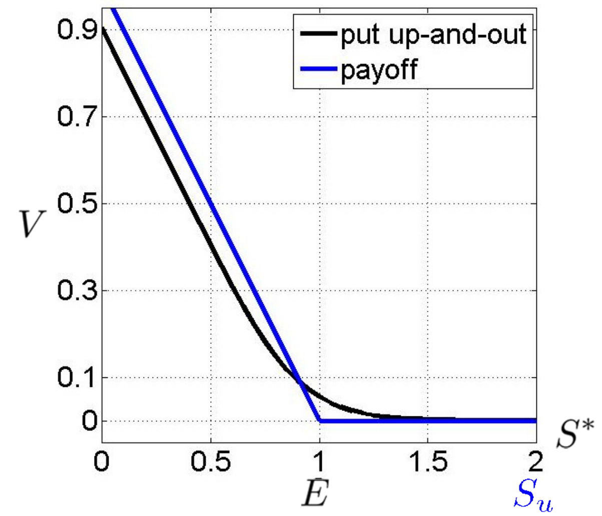
[L.V. Ballestra – G. Pacelli, 2014]

$\sigma = 0.25$ constant volatility
$E = 1$ exercise price
$r = 0.1$ interest rate
$\delta = 0$ dividend yield
$T = 1$ maturity

$e^{x^*} = S^* = [0 : 0.05 : 2]$
current underlying asset values

$S_u = 2$ upper barrier $> E$

$V(S^*, 0)$



FINITE DIFFERENCES

$\Delta t = \Delta x^2$ (implicit in time and centered in space)

Δx	*Max Abs Err*	*Max Rel Err*	CPU *time*
0.1	$3.2\ 10^{-3}$	$4.5\ 10^{+0}$	$1.6\ 10^{-2}$ s
0.05	$9.2\ 10^{-4}$	$1.2\ 10^{+0}$	$1.6\ 10^{-2}$ s
0.025	$2.5\ 10^{-4}$	$9.6\ 10^{-2}$	$1.1\ 10^{-1}$ s
0.0125	$6.6\ 10^{-5}$	$2.5\ 10^{-2}$	$2.3\ 10^{+0}$ s
0.00625	$1.4\ 10^{-5}$	$5.9\ 10^{-3}$	$5.8\ 10^{+1}$ s
0.003125	$3.6\ 10^{-6}$	$1.6\ 10^{-3}$	$1.9\ 10^{+3}$ s

SABO

Δt	*Max Abs Err*	*Max Rel Err*	CPU *time*
0.1	$2.7\ 10^{-6}$	$5.6\ 10^{-2}$	$7.0\ 10^{-1}$ s
0.05	$7.2\ 10^{-7}$	$1.5\ 10^{-2}$	$1.5\ 10^{+0}$ s
0.025	$1.8\ 10^{-7}$	$3.8\ 10^{-3}$	$2.7\ 10^{+0}$ s
0.0125	$4.9\ 10^{-8}$	$1.0\ 10^{-3}$	$5.5\ 10^{+0}$ s
0.00625	$1.6\ 10^{-8}$	$3.4\ 10^{-4}$	$1.1\ 10^{+1}$ s
0.003125	$4.9\ 10^{-9}$	$9.8\ 10^{-5}$	$2.3\ 10^{+1}$ s

[C. Guardasoni - S. Sanfelici, *A boundary element approach to barrier option pricing in Black–Scholes framework*, International Journal of Computer Mathematics, 2016]

Numerical Example: test with constant parameters

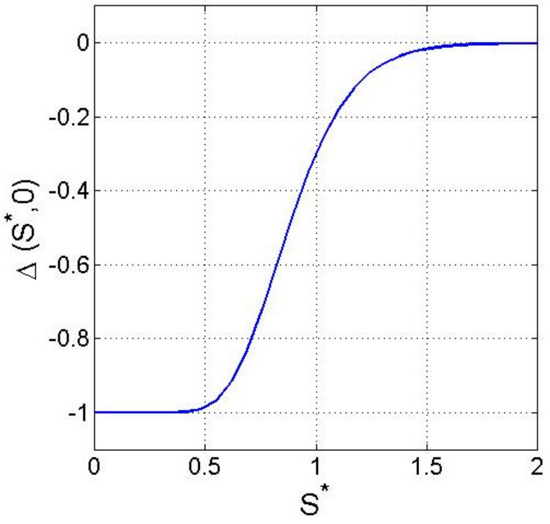
Δ - Hedging

$\sigma = 0.25$ constant volatility
$E = 1$ exercise price
$r = 0.1$ interest rate
$\delta = 0$ dividend yield
$T = 1$ maturity

$e^{x^*} = S^*$
current underlying asset values

$S_u = 2$ upper barrier $> E$

$$\Delta(S^*, 0)$$



approximation by 2^{nd} order CENTERED FINITE DIFFERENCE and closed formula option values

Δx	$S^*=1.9$	$S^*=1$
0.1	$-1.330647\ 10^{-3}$	$-3.068228\ 10^{-1}$
0.05	$-1.259514\ 10^{-3}$	$-3.015779\ 10^{-1}$
0.025	$-1.242075\ 10^{-3}$	$-3.002400\ 10^{-1}$
0.0125	$-1.237736\ 10^{-3}$	$-2.999038\ 10^{-1}$
0.00625	$-1.236653\ 10^{-3}$	$-2.998197\ 10^{-1}$
0.003125	$-1.236382\ 10^{-3}$	$-2.997986\ 10^{-1}$

SABO

Δt	$S^*=1.9$	$S^*=1$
0.1	$-1.217824\ 10^{-3}$	$-2.997916\ 10^{-1}$
0.05	$-1.232680\ 10^{-3}$	$-2.997916\ 10^{-1}$
0.025	$-1.235895\ 10^{-3}$	$-2.997916\ 10^{-1}$
0.0125	$-1.236180\ 10^{-3}$	$-2.997916\ 10^{-1}$
0.00625	$-1.236224\ 10^{-3}$	$-2.997916\ 10^{-1}$
0.003125	$-1.236268\ 10^{-3}$	$-2.997916\ 10^{-1}$

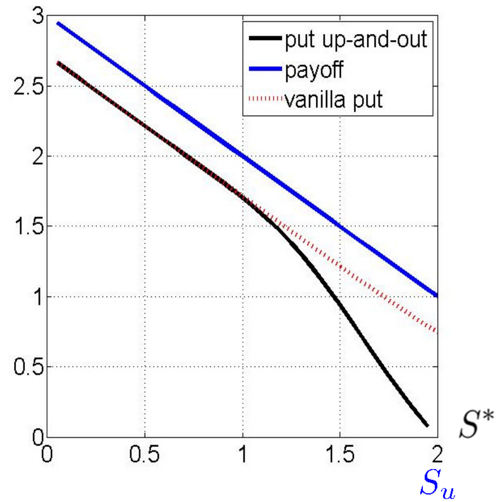
N.B.: in the case of constant parameters, we compare results with the closed formula for the greek.

Numerical Example: test with constant parameters

$E = 3$ exercise price
$e^{x^*} = S^* = [0 : 0.05 : 2]$
current underlying asset values

$S_u = 2$ upper barrier $< E$

$V(S^*, 0)$



SABO

	Max Abs Err	*Max Rel Err*	*CPU time*
0.1	$7.4\ 10^{-4}$	$9.6\ 10^{-3}$	$7.8\ 10^{-1}$ s
0.05	$2.0\ 10^{-4}$	$2.6\ 10^{-3}$	$1.4\ 10^{+0}$ s
0.025	$5.2\ 10^{-5}$	$6.8\ 10^{-4}$	$2.5\ 10^{+0}$ s
0.0125	$1.5\ 10^{-5}$	$1.9\ 10^{-4}$	$4.9\ 10^{+0}$ s
0.00625	$5.3\ 10^{-6}$	$6.4\ 10^{-5}$	$9.7\ 10^{+0}$ s

FINITE DIFFERENCES

$\Delta t = \Delta x^2$ (implicit in time and centered in space)

	Max Abs Err	*Max Rel Err*	CPU *time*
0.1	$2.6\ 10^{-1}$	$3.3\ 10^{+0}$	$1.6\ 10^{-2}$ s
0.05	$8.6\ 10^{-2}$	$1.0\ 10^{+0}$	$1.6\ 10^{-2}$ s
0.025	$3.1\ 10^{-4}$	$8.1\ 10^{-3}$	$1.3\ 10^{-1}$ s
0.0125	$8.1\ 10^{-4}$	$2.0\ 10^{-3}$	$2.4\ 10^{+0}$ s
0.00625	$2.0\ 10^{-4}$	$4.9\ 10^{-4}$	$6.1\ 10^{+1}$ s

MONTE CARLO

$M = 50\,000$ is the initial sampling
$N_{\Delta t} = 100$ is the number of initial time interval decomposition

$(M, N_{\Delta t}) \cdot k$	*Max Abs Err*	*Max Rel Err*	CPU *time*
k=1	$5.0\ 10^{-2}$	$5.7\ 10^{-1}$	$5.1\ 10^{+0}$ s
k=2	$3.4\ 10^{-2}$	$4.4\ 10^{-1}$	$2.7\ 10^{+1}$ s
k=3	$2.7\ 10^{-2}$	$3.2\ 10^{-1}$	$7.2\ 10^{+1}$ s

[C. Guardasoni - S. Sanfelici, *A boundary element approach to barrier option pricing in Black–Scholes framework*, International Journal of Computer Mathematics, 2016]

Numerical Example

[F. Zirilli, L. Fatone, M.C. Recchioni, (2008)]

piecewise constant volatility

$$S_u = 101 \qquad t_0 = 0\,,\ T = 0.5 \qquad S^* = 100 \qquad r = 0.03, \qquad d = 0.02,$$

$$\sigma(t) = \begin{cases} 0.0105 & t < 0.25 \\ 0.01147824 & 0.25 \leq t \leq T \end{cases}$$

$E = 101$

n	$V_{SABO}(100,0)$	CPU time (s)
2	0.89178	$1.0 \cdot 10^{+0}$
3	0.89373	$2.0 \cdot 10^{+0}$
4	0.89419	$4.5 \cdot 10^{+0}$
5	0.89433	$1.3 \cdot 10^{+1}$
6	0.89436	$3.9 \cdot 10^{+1}$
7	0.89437	$1.2 \cdot 10^{+2}$

n	$V_{FD}(100,0)$	CPU time (s)
0	0.89584	$1.0 \cdot 10^{-1}$
1	0.89474	$2.1 \cdot 10^{+0}$
2	0.89447	$3.4 \cdot 10^{+1}$
3	0.89440	$3.4 \cdot 10^{+2}$
4	0.89438	$3.4 \cdot 10^{+3}$

$$\Delta t_{SABO} = T/2^n$$

$E = 103$

$$\Delta t_{FD} = \Delta x_{FD}^2 \qquad \Delta x_{FD} = 0.25/2^n$$

n	$V_{SABO}(100,0)$	CPU time (s)
2	1.08163	$1.0 \cdot 10^{+0}$
3	1.08634	$1.9 \cdot 10^{+0}$
4	1.08787	$4.5 \cdot 10^{+0}$
5	1.08828	$1.2 \cdot 10^{+1}$
6	1.08839	$3.7 \cdot 10^{+1}$
7	1.08842	$1.2 \cdot 10^{+2}$

n	$V_{FD}(100,0)$	CPU time (s)
0	1.10233	$1.6 \cdot 10^{-1}$
1	1.09175	$2.4 \cdot 10^{+0}$
2	1.08926	$3.5 \cdot 10^{+1}$
3	1.08864	$3.7 \cdot 10^{+2}$
4	1.08849	$3.7 \cdot 10^{+3}$

[C. Guardasoni, *Semi-Analytical method for the pricing of barrier options in case of time-dependent parameters (with Matlab codes*), submitted to CAIM]

Numerical Example

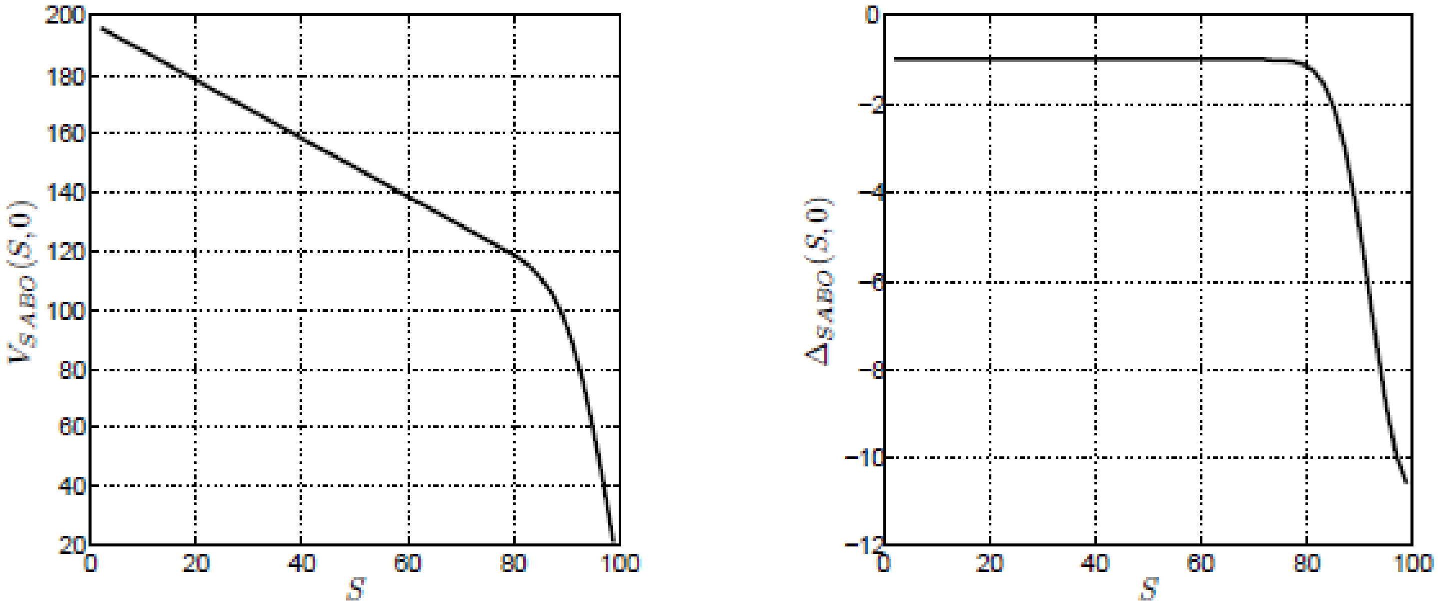
[F. Zirilli, L. Fatone, M.C. Recchioni, (2008)]

piecewise constant volatility

$$S_u = 101 \qquad t_0 = 0\,,\ T = 0.5 \qquad S^* = 100 \qquad r = 0.03, \qquad d = 0.02,$$

$$\sigma(t) = \begin{cases} 0.0105 & t < 0.25 \\ 0.01147824 & 0.25 \leq t \leq T \end{cases}$$

$E = 201 \qquad \Delta t = 0.0625$



[C. Guardasoni, *Semi-Analytical method for the pricing of barrier options in case of time-dependent parameters (with Matlab codes)*, submitted to CAIM]

Numerical Example

[F. Zirilli, L. Fatone, M.C. Recchioni, (2008)]

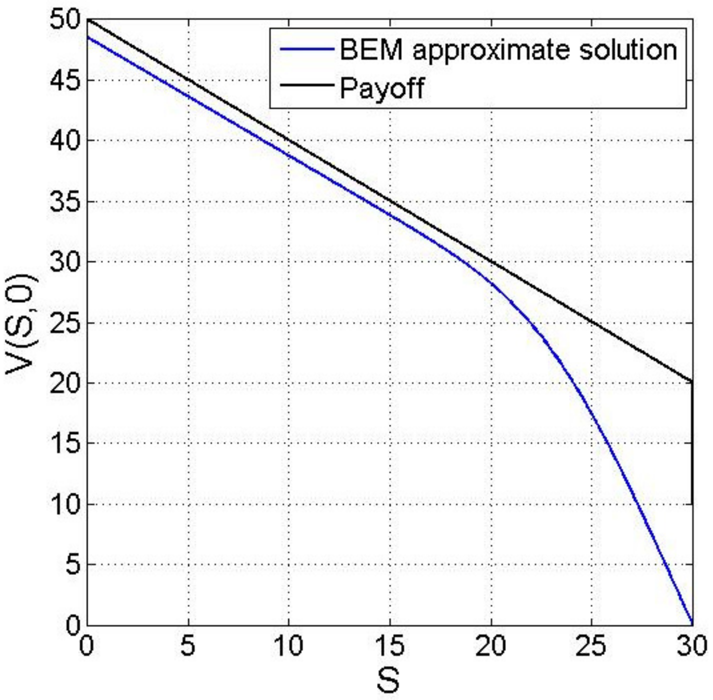
time-continuous volatility

$S_u = 30 \qquad t_0 = 0\,,\ T = 1 \qquad S^* = 29 \qquad r = 0.03, \qquad d = 0.02, \qquad E = 50$

$\sigma^2(t) = 0.03 + 0.02(T - t)$

$\Delta t = T/2^n$ **SABO**

n	$V(S^*, 0)$	*CPU time*
4	3.67754	4.0 10^{-0}
5	3.68136	1.0 10^{+1}
6	3.68235	3.4 10^{+1}
7	3.68264	1.2 10^{+2}



[C. Guardasoni, *Semi-Analytical method for the pricing of barrier options in case of time-dependent parameters (with Matlab codes*), submitted to CAIM]

Numerical Example

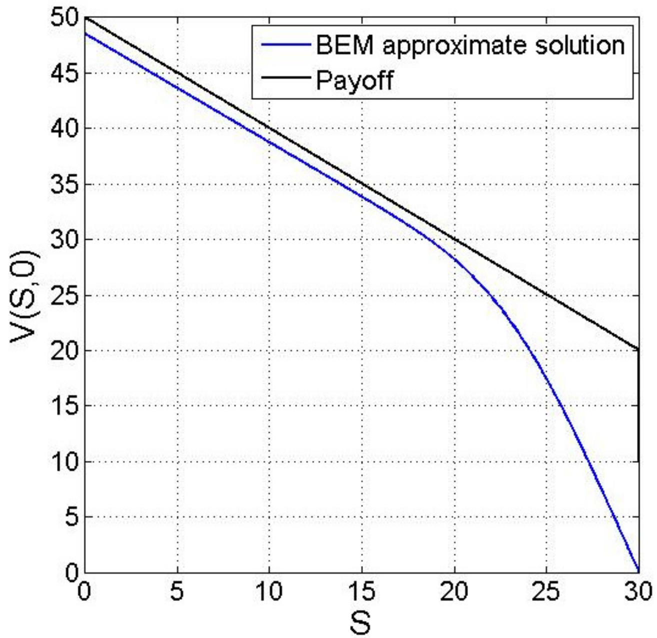
[F. Zirilli, L. Fatone, M.C. Recchioni, (2008)]

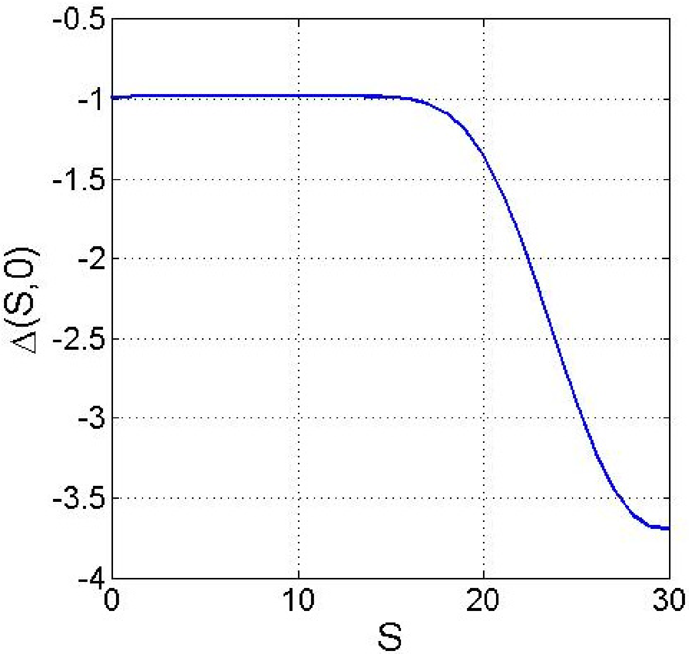
$S_u = 30 \qquad t_0 = 0\,,\ T = 1 \qquad S^* = 29 \qquad r = 0.03, \qquad d = 0.02, \qquad E = 50$

$\sigma^2(t) = 0.03 + 0.02(T - t)$

$\Delta t = 0.05$

Δ - Hedging





[C. Guardasoni, *Semi-Analytical method for the pricing of barrier options in case of time-dependent parameters (with Matlab codes*), submitted to CAIM]

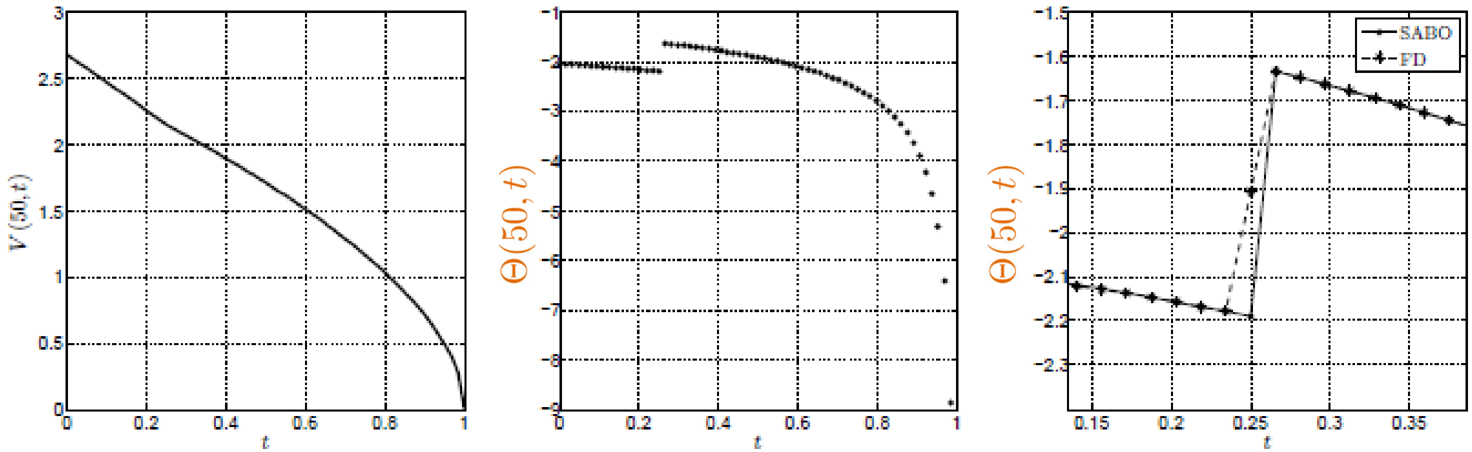
Numerical Example

[F. Zirilli, L. Fatone, M.C. Recchioni, (2008)]

$S_u = 101 \quad E = 50 \quad t_0 = 0\,, \; T = 1 \quad S^* = 50 \quad \sigma = 0.105, \quad d = 0.05,$

$$r(t) = \begin{cases} 0.01 & t < 0.25 \\ 0.03 & 0.25 \le t \le T \end{cases}$$

$\Delta t = 0.015625$ $\quad\quad$ $\Delta t = \Delta x^2 \quad \Delta x = 0.125$



$$\Theta(S,t) := \frac{\partial V}{\partial t}(S,t) = -\frac{\partial u}{\partial \tau}(log(S), T-t)e^{-\int_t^T \bar{r}(t')dt'} + V(S,t)r(t)$$

$$\frac{\partial u}{\partial \tau}(x,\tau) := \int_{-\infty}^{U} u_0(y)\frac{\partial G}{\partial \tau}(y,0;x,\tau)dy + \int_0^{\tau} \frac{\overline{\sigma}^2(s)}{2}\frac{\partial G}{\partial \tau}(U,s;x,\tau)\frac{\partial u}{\partial y}(U,s)ds$$

$$\frac{\partial G}{\partial \tau}(y,s,x,\tau) = \frac{G(y,s,x,\tau)}{\int_s^{\tau}\overline{\sigma}^2(v)dv}\left\{\left[y - x - \int_s^{\tau}(\overline{r} - \frac{\overline{\sigma}^2}{2} - \overline{d})(v)dv\right](\overline{r} - \frac{\overline{\sigma}^2}{2} - \overline{d})(\tau) + \right.$$
$$\left. + \left(\frac{\left[y - x - \int_s^{\tau}(\overline{r} - \frac{\overline{\sigma}^2}{2} - \overline{d})(v)dv\right]^2}{\int_s^{\tau}\overline{\sigma}^2(v)dv} - 1\right)\frac{\overline{\sigma}^2(\tau)}{2}\right\}.$$

[C. Guardasoni, ... CAIM]

Observations

Advantages :

- implicit satisfaction of asset infinity boundary conditions
- avoidance of discretization of asset- domain (dimensional reduction)
- high precision and stability
- direct evaluation of derivated functions (greeks)

Costs are due to:

- discretization in time
- numerical quadrature

Needs :

- Green fundamental solution in a closed or approximated form

Application to Heston model

[S.L. Heston (1993)]

if the **volatility** is considered as a **stochastic process** the problem to evaluate a **DOWN-and-OUT Call Option** reduces to the following partial differential problem

$\rho =$ correlation between S and v
$\eta =$ volatility of volatility
$\lambda =$ speed of mean reversion
$\theta =$ long-run variance
$r =$ risk-free interest rate
$\delta =$ dividend yield

V depends also on the square of volatility v

$V(x, v, t)$ option price $\quad x \in \Omega_x = (\log(L), +\infty)$, $v \in \Omega_v = (0, +\infty)$, $t \in [0, T)$

$$\frac{\partial V}{\partial t} + \frac{1}{2} v \frac{\partial^2 V}{\partial x^2} + \rho \eta v \frac{\partial^2 V}{\partial x \partial v} + \frac{1}{2} \eta^2 v \frac{\partial^2 V}{\partial v^2} + \left(r - \delta - \frac{1}{2} v \right) \frac{\partial V}{\partial x} - (\lambda(v - \bar{v}) - \theta v) \frac{\partial V}{\partial v} - rV = 0$$

Application to Heston model

[S.L. Heston (1993)]

if the **volatility** is considered as a **stochastic process** the problem to evaluate a **DOWN-and-OUT Call Option** reduces to the following partial differential problem

$\rho =$ correlation between S and v
$\eta =$ volatility of volatility
$\lambda =$ speed of mean reversion
$\theta =$ long-run variance
$r =$ risk-free interest rate
$\delta =$ dividend yield

V depends also on the square of volatility v

$V(x,v,t)$ option price $\quad x \in \Omega_x = (\log(L), +\infty),\ v \in \Omega_v = (0, +\infty),\ t \in [0,T)$

$$\frac{\partial V}{\partial t} + \frac{1}{2} v \frac{\partial^2 V}{\partial x^2} + \rho \eta v \frac{\partial^2 V}{\partial x \partial v} + \frac{1}{2}\eta^2 v \frac{\partial^2 V}{\partial v^2} + \left(r - \delta - \frac{1}{2} v\right)\frac{\partial V}{\partial x} - (\lambda(v - \bar{v}) - \theta v)\frac{\partial V}{\partial v} - rV = 0$$

- **final condition (payoff)** $\quad V(x,v,T) = \max(e^x - E, 0) \quad x \in \Omega_x \quad v \in \Omega_v$
  with E exercise price

boundary conditions **[E. Miglio-C. Sgarra (2011)]**

- on the asset

$$V(\log(L), v, t) = 0 \qquad \lim_{x \to -\infty} V(x,v,t) = e^{x - \delta t} \qquad t \in [0,T) \quad v \in \Omega_v$$

- on the variance

$$\lim_{v \to +\infty} S(x,v,t) = e^x \qquad S(x,0,t) = \sum_{n=0}^{+\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} S_{BS}(t, e^x, B, \bar{\sigma}_n, \bar{r}_n) \qquad x \in \Omega_x \quad t \in [0,T)$$

$S_{BS}(t, e^x, B, \bar{\sigma}_n, \bar{r}_n)$ Black-Scholes value with
variance $\bar{\sigma}_n^2 = \dfrac{n\sigma^2}{t}$ and rate $\bar{r}_n = r - \delta + \lambda(1 - e^{\mu + \sigma^2/2}) + n\dfrac{\mu + \sigma^2/2}{t}$

Numerical methods

Analitical solution only for Vanilla option [P.Carr, D.Madan (1999)]

NO ANALYTICAL SOLUTION

Finite Elements

Finite Differences

Monte Carlo

<u>Shortcomings</u>

- evaluation of the option over a wide subset of asset and variance values
- approximation of boundary conditions due to the truncation of the domain

<u>Shortcoming</u>

- slow convergence

representation formula

[C.Guardasoni, S.Sanfelici (2016)]

$$x \in \Omega_x = (\log(L), +\infty),\ v \in \Omega_v = (0, +\infty),\ t \in [0, T)$$

$$V(x,v,t) = e^{-r(T-t)} \left\{ \int_{\log(L)}^{+\infty} \int_{\Omega_v} V(y,w,T) G(x,y,v,w,t,T)\, dw\, dy + \right.$$
$$\left. - \int_t^T \int_{\Omega_v} \frac{\partial V}{\partial y}(\log(L), w, \tau) e^{r(T-t)} \frac{w}{2} G(x, \log(L), v, w, t, \tau) dw\, d\tau \right\}$$

Fundamental solution

[C. Guardasoni, S. Sanfelici (SIAM 2016)]

$G(x,y,v,w,t,\tau)$ is the joint transition probability density (or **fundamental solution**) that expresses the probability to move from (x,v) at time t to (y,w) at time τ

$$G(x,y,v,w,t,\tau) = p_{t\to\tau}(x\to y, v\to w) = p_{t\to\tau}(y-x,w|v) = p_{t\to\tau}(y-x|w,v)\widetilde{p}_{t\to\tau}(v,w)$$

- $\widetilde{p}_{t\to\tau}(v,w)$ is the transition density of the variance v conditioned on w **[W. Feller (1951)]**

$$\widetilde{p}_{t\to\tau}(v,w) = \gamma e^{-\gamma\left(ve^{-\lambda(\tau-t)}+w\right)}\left(\frac{w}{ve^{-\lambda(\tau-t)}}\right)^{\frac{\alpha-1}{2}} I_{\alpha-1}\left(2\sqrt{\gamma^2 vwe^{-\lambda(\tau-t)}}\right)$$

$$\gamma = \frac{2\lambda}{\left(1-e^{-\lambda(\tau-t)}\right)\eta^2} \qquad \alpha = \frac{2\lambda\bar{v}}{\eta^2};$$

I is the modified Bessel function of the 1st kind (Feller condition $\lambda\bar{v}\geq\eta^2$)

- with an inverse Fourier transform:

$$p_{t\to\tau}(y-x|w,v) = \mathcal{F}_{\omega}^{-1}[\widehat{p}](y-x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{p}(\omega,v,w,t,\tau)e^{-\mathbf{i}\omega(y-x)}d\omega$$

$$\widehat{p}(\omega,v,w,t,\tau) = e^{\mathbf{i}\omega\left\{(r-d)(\tau-t)+\frac{\rho}{\eta}\left(w-v-\lambda\bar{v}(\tau-t)\right)\right\}}\phi\left[\omega\left(\frac{\lambda\rho}{\eta}-\frac{1}{2}\right)+\frac{1}{2}\mathbf{i}\omega^2(1-\rho^2)\right]$$

$\phi[\cdot] = \ldots$ is the characteristic function of the integrated variance $\int_t^\tau v(s)ds$ given v_t and v_τ

[M. Broadie, O. Kaya (2006)]

Numerical resolution of BIE

[C. Guardasoni, S. Sanfelici (SIAM 2016)]

- uniform decomposition of the time interval $[0,T]$ with time step

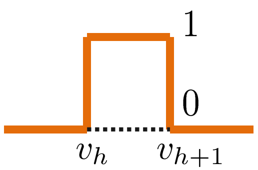
$$\Delta t = T/N_{\Delta t}: \qquad t_j = j\Delta t \quad j = 0,\dots,N_{\Delta t}$$

- uniform decomposition of the variance interval $[0, v_{\max}]$ with step

$$\Delta v = v_{\max}/N_{\Delta v}: \qquad v_i = i\Delta v \quad i = 0,\dots,N_{\Delta v}$$

- approximation of the BIE unknown

$$q(\log(L), w, \tau) \approx \sum_{h=1}^{N_{\Delta v}} \sum_{k=1}^{N_{\Delta t}} \alpha_h^{(k)} \psi_h(w) \varphi_k(\tau)$$



with
$$\psi_h(w) = H[w - v_{h-1}] - H[w - v_h] \qquad \text{for} \qquad h = 1,\dots,N_{\Delta v}$$
$$\varphi_k(\tau) = H[\tau - t_{k-1}] - H[\tau - t_k] \qquad\qquad k = 1,\dots,N_{\Delta t}$$

- evaluation of BIE at the collocation nodes:

$$\bar{t}_j = \frac{t_j + t_{j-1}}{2} \qquad j = 1,\dots,N_{\Delta t}$$
$$\bar{v}_i = \frac{v_i + v_{i-1}}{2} \qquad i = 1,\dots,N_{\Delta v}$$

Attention!: the fundamental solution in this framework is known throughout a numerical inverse Fourier transform

Numerical resolution of BIE

[C. Guardasoni, S. Sanfelici (SIAM 2016)]

$$\boxed{\mathcal{A}\alpha = \mathcal{F}}$$

- $\mathcal{A}$ has an upper triangular Toeplitz structure $\qquad \ell = k - j \quad \ell = 0, \dots, N_{\Delta t}$, $\quad i, h = 1, \dots, N_{\Delta v}$

$$\mathcal{A}_{ih}^{(jk)} = \int_{\max(\bar{t}_j, t_{k-1})}^{t_k} \int_{v_{h-1}}^{v_h} \frac{w}{2} G(\log(L), \log(L), \bar{v}_i, w, \bar{t}_j, \tau)\, dw\, d\tau =$$

$$= \int_{\frac{1}{2}-\frac{1}{2}H[\ell]}^{1} \int_{v_{h-1}}^{v_h} \frac{\Delta t}{4\pi} w \widetilde{p}_{0\to\Delta t(\ell-\frac{1}{2}+s)}(\bar{v}_i, w) \int_{-\infty}^{+\infty} \widehat{p}(\omega, \bar{v}_i, w, 0, \Delta t(\ell - \frac{1}{2} + s))\, d\omega\, dw\, ds =: \mathcal{A}_{ih}^{(\ell)}$$

$$\begin{pmatrix} \mathcal{A}^{(0)} & \mathcal{A}^{(1)} & \cdots & \mathcal{A}^{(N_{\Delta t}-2)} & \mathcal{A}^{(N_{\Delta t}-1)} \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & \mathcal{A}^{(0)} & \mathcal{A}^{(1)} & \mathcal{A}^{(2)} \\ 0 & \cdots & 0 & \mathcal{A}^{(0)} & \mathcal{A}^{(1)} \\ 0 & \cdots & 0 & 0 & \mathcal{A}^{(0)} \end{pmatrix} \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \vdots \\ \alpha^{(N_{\Delta t}-1)} \\ \alpha^{(N_{\Delta t})} \end{pmatrix} = \begin{pmatrix} \mathcal{F}^{(1)} \\ \mathcal{F}^{(2)} \\ \vdots \\ \mathcal{F}^{(N_{\Delta t}-1)} \\ \mathcal{F}^{(N_{\Delta t})} \end{pmatrix}$$

$$\mathcal{A}^{(0)}\,\alpha^{(\ell)} = \mathcal{F}^{(\ell)} - \sum_{q=\ell+1}^{N_{\Delta t}} \mathcal{A}^{(q-\ell)}\,\alpha^{(q)}, \quad \ell = 1, \cdots, N_{\Delta t}$$

- numerical quadrature rule for evaluation of inverse Fourier transform: Matlab adaptive quadrature
- numerical quadrature rule for evaluation of integrals: Gauss-Legendre quadrature rules

Numerical example: Heston model

L.Feng-V.Linetsky (2008)

$E = 100$ exercise price
$r = 0.05$ interest rate
$\delta = 0.02$ asset payout ratio
$\rho = -0.5$ correlation between S and v
$\eta = 0.1$ volatility of volatility
$\lambda = 4$ speed of mean reversion
$\bar{v} = 0.04$ long-run variance

$L = 110$ barrier

$e^{x^*} = S^*$ current underlying asset value
$v^* = 0.01$ current variance
$T = 1$ maturity

$V(150, v^*, 0)$

SABO

$N_{\Delta t} = N_{\Delta v}$	$V(S^*, v^*, 0)$	CPU time
3	50.96	$2 \cdot 10^{+2}$ s
6	50.98	$9 \cdot 10^{+2}$ s
9	51.02	$2 \cdot 10^{+3}$ s
12	51.01	$2 \cdot 10^{+4}$ s
15	51.01	$4 \cdot 10^{+4}$ s

MONTE CARLO

sampling

time discretization

	$M = 10^4$			$M = 10^6$			$M = 10^8$		
$N_{\Delta t} = 100$	51.49	[50.95 , 52.02]	$4 \cdot 10^{-1}$	51.24	[51.18 , 51.29]	$4 \cdot 10^{+1}$	51.25	[51.25 , 51.26]	$4 \cdot 10^{+3}$
$2\,N_{\Delta t}$	51.24	[50.71 , 51.78]	$6 \cdot 10^{-1}$	51.19	[51.14 , 51.25]	$6 \cdot 10^{+1}$	51.19	[51.18 , 51.19]	$7 \cdot 10^{+3}$
$4\,N_{\Delta t}$	51.33	[50.79 , 51.88]	$1 \cdot 10^{+0}$	51.16	[51.10 , 51.21]	$1 \cdot 10^{+2}$	51.14	[51.13 , 51.15]	$1 \cdot 10^{+4}$
$8\,N_{\Delta t}$	51.35	[50.80 , 51.89]	$2 \cdot 10^{+0}$	51.13	[51.08 , 51.18]	$2 \cdot 10^{+2}$	51.11	[51.10 , 51.11]	$2 \cdot 10^{+4}$
$16\,N_{\Delta t}$	51.41	[50.88 , 51.95]	$4 \cdot 10^{+0}$	51.11	[51.05 , 51.16]	$3 \cdot 10^{+2}$	51.09	[51.08,51.09]	$5 \cdot 10^{+4}$
$32\,N_{\Delta t}$	50.97	[50.42 , 51.52]	$8 \cdot 10^{+0}$	51.08	[51.02 , 51.13]	$1 \cdot 10^{+3}$	51.07	[51.06,51.07]	$7 \cdot 10^{+4}$

Numerical example: Heston model

$V(115, v^*, 0)$

SABO

$N_{\Delta t} = N_{\Delta v}$	$V(S^*, v^*, 0)$
3	8.04
6	8.06
9	8.31
12	8.29
15	8.30

CPU time

BEM, $S = 115$

pre- and postprocessing

$N_{\Delta t} = N_{\Delta v}$	Times
3	1.5E+02 s
6	7.5E+02 s
9	3.4E+03 s
12	3.7E+03 s
15	6.2E+03 s

postprocessing

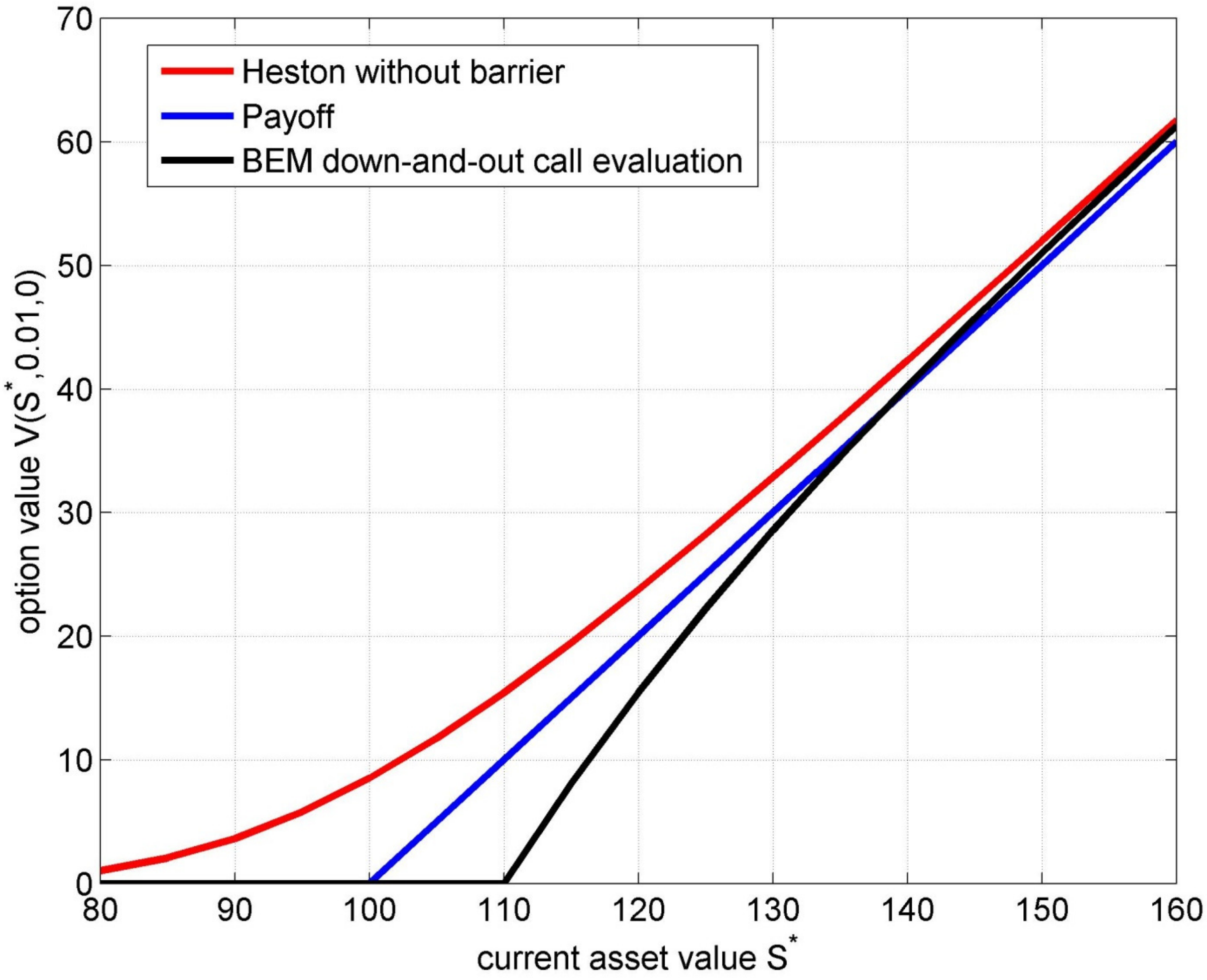
$N_{\Delta t} = N_{\Delta v}$	Times
3	3.8E+01 s
6	1.4E+02 s
9	3.1E+02 s
12	3.9E+02 s
15	6.1E+02 s

MONTE CARLO

sampling (columns); time discretization (rows)

	$M = 10^4$			$M = 10^6$			$M = 10^8$		
$N_{\Delta t} = 100$	9.86	[9.51,10.21]	$4 \cdot 10^{-1}$	9.72	[9.69,9.76]	$4 \cdot 10^{+1}$	9.74	[9.73,9.74]	$4 \cdot 10^{+3}$
$2\,N_{\Delta t}$	9.22	[8.88,9.57]	$7 \cdot 10^{-1}$	9.33	[9.30,9.37]	$7 \cdot 10^{+1}$	9.33	[9.32,9.33]	$6 \cdot 10^{+3}$
$4\,N_{\Delta t}$	8.99	[8.64,9.33]	$1 \cdot 10^{+0}$	9.04	[9.00,9.07]	$1 \cdot 10^{+2}$	9.03	[9.03,9.04]	$1 \cdot 10^{+4}$
$8\,N_{\Delta t}$	8.81	[8.46,9.15]	$2 \cdot 10^{+0}$	8.82	[8.79,8.86]	$2 \cdot 10^{+2}$	8.83	[8.83, 8.83]	$2 \cdot 10^{+4}$
$16\,N_{\Delta t}$	8.54	[8.20,8.88]	$4 \cdot 10^{+0}$	8.68	[8.65,8.71]	$4 \cdot 10^{+2}$	8.68	[8.68, 8.68]	$4 \cdot 10^{+4}$
$32\,N_{\Delta t}$				8.58	[8.54,8.61]	$8 \cdot 10^{+2}$			
$64\,N_{\Delta t}$				8.53	[8.49,8.56]	$2 \cdot 10^{+3}$			
$128\,N_{\Delta t}$				8.50	[8.46,8.53]	$3 \cdot 10^{+3}$			
$256\,N_{\Delta t}$				8.43	[8.40,8.47]	$7 \cdot 10^{+3}$			
$512\,N_{\Delta t}$				8.39	[8.36,8.43]	$1 \cdot 10^{+4}$			

Numerical example: Heston model



References

Semi-Analytical method for the pricing of Barrier Options:

- *A boundary element approach to barrier option pricing in* ***Black–Scholes framework***

  International Journal of Computer Mathematics, 2016

- *Fast numerical pricing of barrier options under* ***stochastic volatility and jumps***

  SIAM Journal on Applied Mathematics, 2016

- *Semi-Analytical method for the pricing of barrier options in case of* ***time-dependent parameters*** *(with Matlab codes)*

  submitted to CAIM

Perspective

- **Extension to Asian barrier options with geometric mean**
- **… with arithmetic mean**

Thank you for the attention!

C. Guardasoni, S. Sanfelici
University of Parma, Italy

UNIMORE
UNIVERSITÀ DEGLI STUDI DI MODENA E REGGIO EMILIA

Kolmogorov-Fokker-Planck Equations:
theoretical issues and applications

April 10 - 11, 2017
Modena