Well-posedness of semilinear stochastic wave equations with Hölder continuous coefficients

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We prove strong well-posedness for the semilinear stochastic wave equation

$$\begin{cases} \frac{\partial^2}{\partial \tau^2} y(\tau, \xi) = \frac{\partial^2}{\partial \xi^2} y(\tau, \xi) + b(\tau, \xi, y(\tau, \xi)) + \epsilon \dot{W}(\tau, \xi), \\ \\ y(\tau, 0) = y(\tau, 1) = 0, \\ y(0, \xi) = x_0(\xi), \\ \frac{\partial y}{\partial \tau}(0, \xi) = x_1(\xi), \quad \tau \in (0, T], \quad \xi \in [0, 1], \end{cases} \tag{1}$$

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space; $\epsilon \neq 0$;
- $\dot{W}(\tau, \xi)$ space-time white noise, i.e., formally the time-derivative of a cylindrical Wiener process

$$W(\tau, \xi) = \sum_{k \geqslant 1} \beta_k(\tau) e_k(\xi);$$

- $x_0 \in H_0^1([0, 1])$ $x_1 \in L^2([0, 1])$; moreover (e_k) is a basis of eigenfunctions in $L^2([0, 1])$
- b bounded measurable, α-Hölder continuous in y, $\alpha \in (2/3, 1)$.
- Without noise (i.e. $\epsilon = 0$) for equation (4) there is no uniqueness in general.

PLAN

1. Setting of the problem (*we concentrate on semilinear stochastic wave equations but also semilinear stochastic plate equations can be treated*)

2. Overview of regularization by noise for ODEs

3. Some recent results on regularization by noise for PDEs

4. Stochastic wave equation

5. Main result

6. Some ideas about the proof

A novelty in our approach is the use of FBSDEs (forward-backward stochastic differential equations)

SEMINAL PAPERS

- A.K. Zvonkin : Mat. Sb. (N.S.) (1974) [$b \in L^\infty(\mathbb{R})$, i.e., $d = 1$]

- A.J. Veretennikov : Mat. Sb., (N.S.) (1980) [$b \in L^\infty(\mathbb{R}^d)$ (for any $d \geqslant 1$)].

- ...

• A variant of the Zvonkin-Veretennikov approach: the Ito-Tanaka trick for SDEs (cf. Flandoli-Gubinelli-P. 2010) :

To simplify $\mathbb{R}^d = \mathbb{R}$. Let $b : \mathbb{R} \to \mathbb{R}$ be an irregular function (it could be Hölder continuous).

Our equation is

$$\boxed{X_t = x + \int_0^t b(X_s)ds + W_t, \quad t \geqslant 0, \quad x \in \mathbb{R}.}$$

We write

$$X_t - x - W_t = \int_0^t b(X_s)ds.$$

Now if v is a “regular” solution of

$$\lambda v - Lv = b \quad \text{on } \mathbb{R}, \quad \lambda > 0,$$

$L = \frac{1}{2}\frac{d^2}{dx^2} + b(x) \cdot \frac{d}{dx}$ then by Itô’s formula:

$$v(X_t) = v(x) + \int_0^t v'(X_s)dW_s + \int_0^t Lv(X_s)ds$$

and so

$$v(X_t) = v(x) + \int_0^t v'(X_s)dW_s + \int_0^t (\lambda v(X_s) - b(X_s))ds$$

and

$$X_t + v(X_t) = x + v(x) + W_t + \int_0^t v'(X_s)dW_s + \lambda \int_0^t v(X_s)ds$$

$\Rightarrow$ **uniqueness thank to the regularity of** v

Regularization by additive noise for parabolic PDEs

Gyöngy I., Pardoux E., "On the regularization effect of space-time white noise on quasi-linear parabolic partial differential equations", Probability Theory and Related Fields 1993

Da Prato G., Flandoli F., "Pathwise uniqueness for a class of SDEs in Hilbert spaces and applications." J. Funct. Anal. 2010

Da Prato G., Flandoli F., Priola E., Röckner M. "Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift." Ann. Probab. 2013

Wang F. Y., Zhang X., Degenerate SDEs in Hilbert spaces with rough drifts. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 2015

Mytnik L., Neuman E., Pathwise uniqueness for the stochastic heat equation with Hölder continuous drift and noise coefficients. Stochastic Process. Appl. 2015

Da Prato G., Flandoli F., Röckner M. and Veretennikov A., Strong uniqueness for SDEs in Hilbert spaces with non-regular drift", Ann. Probab. 2016

Regularization of PDEs by multiplicative noise

- Flandoli F., Gubinelli M., Priola E., Well-posedness of the transport equation by stochastic perturbation. Invent. Math. 180 (2010)

"$d_t u(t,x) + (b(t,x)\cdot Du(t,x))dt + \sum_{i=1}^{d} e_i \cdot Du(t,x) \circ dW_t^i = 0$"

- Fedrizzi E. and Flandoli F., Noise prevents singularities in linear transport equations, J. Funct. Anal. 264, (2013)

.....

- Gess B., Souganidis P. E., Long-Time Behavior, Invariant Measures, and Regularizing Effects for Stochastic Scalar Conservation Laws, to appear in Comm Pure Appl. Math (including stochastic Burgers equations)

$$\text{“}du + \partial_x(u^2) \circ d\beta_t''$$

General formulation

Set $\Lambda = -\dfrac{d^2}{dx^2}$ with Dirichlet boundary conditions is a positive self-adjoint operator with trace class inverse $\Lambda^{-1} : U = L^2([0,1]) \to L^2([0,1])$,

$$\Lambda e_n = n^2\pi^2 e_n, \quad n \geqslant 1, \quad e_n(\xi) = \sqrt{2}\sin(n\pi\xi).$$

$$\mathcal{D}(\Lambda) = H_0^1([0,1]) \cap H^2([0,1]), \; \mathcal{D}(\Lambda^{1/2}) = H_0^1([0,1]), \; \mathcal{D}(\Lambda^{-1/2}) = H^{-1}([0,1]).$$

Thus our initial equation can be written as ($\epsilon = 1$)

$$\begin{cases} \frac{d^2y}{d\tau^2}(\tau) = -\Lambda y(\tau) + B(t, y(\tau), \frac{dy}{d\tau}(\tau)) + \dot{W}(\tau), \\ \\ y(0) = x_0 \in U, \\ \frac{dy}{d\tau}(0) = x_1 \in \mathcal{D}(\Lambda^{-1/2}), \quad \tau \in [0,T], \end{cases} \tag{2}$$

Hypothesis 1. $\Lambda : \mathcal{D}(\Lambda) \subset U \to U$ is a positive self-adjoint operator on a real separable Hilbert space U and there exists Λ^{-1} which is a trace class operator from U into U ($D(\Lambda^{-1/2})$ is the completion of U with respect to $|\Lambda^{-1/2} \cdot|_U$).

Set

$$X_\tau^{0,x} = X_\tau^x = \left(y(\tau), \frac{dy}{d\tau}(\tau)\right)$$

where y is solution to (2), $x = (x_0, x_1)$.

Wave operator: $A = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix}$ generates an unitary group e^{tA} in $H = U \times \mathcal{D}(\Lambda^{-1/2})$.

According to the book [Da Prato-Zabczyk 1992] we study $(\epsilon = 1)$

$$\boxed{\begin{cases} dX_\tau^{0,x} = AX_\tau^{0,x}d\tau + GB(\tau, X_\tau^{0,x})d\tau + GdW_\tau, & \tau \in [0,T], \\ \\ X_0^{0,x} = x \in H = U \times \mathcal{D}(\Lambda^{-1/2}). \end{cases}} \tag{3}$$

where $\boxed{W_\tau = \sum_{n\geqslant 1} \beta_n(\tau)e_n}$, (e_n) is a fixed basis of eigenfunctions in U and $(\beta_n(t))$ are independent real Wiener processes, $G : U \to H$,

$$Gu = \begin{pmatrix} 0 \\ I \end{pmatrix} u, \quad GdW_\tau = \begin{pmatrix} 0 \\ dW_\tau \end{pmatrix}, \; GB(\tau, X_\tau) = \begin{pmatrix} 0 \\ B(\tau, X_\tau) \end{pmatrix}.$$

Recall that for $x = (x_1, x_2) \in H = U \times \mathcal{D}(\Lambda^{-1/2})$, $|x|_H = |x_1|_U + |\Lambda^{-1/2} x_2|_U$.

A solution is a mild solution:

$$\boxed{X_\tau^x = e^{\tau A} x + \int_0^\tau e^{(\tau - s)A} G B(s, X_s) ds + \int_0^\tau e^{(\tau - s)A} G dW_s, \quad \tau \in [0, T] \,.}$$

- $W_A(\tau) := \int_0^\tau e^{(\tau - s)A} G dW_s$ not well defined in the usual $K = \mathcal{D}(\Lambda^{1/2}) \times U$ even if $B = 0$
- X evolves in $H = U \times \mathcal{D}(\Lambda^{-1/2}) = L^2([0, 1]) \times H^{-1}([0, 1])$ even if $B = 0$.
- In the initial example $B(\tau, h) := b(\tau, \xi, h_1(\cdot)) \in L^2([0, 1])$, $h = (h_1, h_2) \in H$, $\xi \in [0, 1]$.

Hypothesis 2. $B : [0, T] \times H \to U$ is Borel, bounded and α-Hölder continuous in x, for some $\alpha \in (2/3, 1)$, i.e. there exists $C = C_\alpha > 0$ such that

$$\sup_{t \in [0, T]} |B(t, x + h) - B(t, x)|_U \leqslant C |h|_H^\alpha, \quad x,\ h \in H.$$

$$\begin{cases} \frac{\partial^2}{\partial \tau^2} y(\tau, \xi) = \frac{\partial^2}{\partial \xi^2} y(\tau, \xi) + b(\tau, \xi, y(\tau, \xi)) + \dot{W}(\tau, \xi), \\ \\ y(\tau, 0) = y(\tau, 1) = 0, \\ y(0, \xi) = x_0(\xi), \\ \frac{\partial y}{\partial \tau}(0, \xi) = x_1(\xi), \quad \tau \in (0, T], \quad \xi \in [0, 1], \end{cases} \tag{4}$$

The function $b : [0, T] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ is measurable and, for $\tau \in [0, T]$ and a.a. $\xi \in [0, 1]$, the map $b(\tau, \xi, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous.

There exists c_1 bounded and measurable on $[0, 1]$, $\alpha \in (2/3, 1)$, such that, for $\tau \in [0, T]$ and for a.a. $\xi \in [0, 1]$,

$$|b(\tau, \xi, x) - b(\tau, \xi, y)| \leqslant c_1(\xi) |x - y|^{\alpha},$$

x, $y \in \mathbb{R}$.

Moreover $|b(\tau, \xi, x)| \leqslant c_2(\xi)$, for $\tau \in [0, T]$ and a.a. $\xi \in [0, 1]$, with $c_2 \in L^2([0, 1])$.

Existence of **weak mild solutions** follows by the Girsanov Theorem

Recall that a (weak) mild solution is a tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, X)$, where $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a stochastic basis on which it is defined a cylindrical U-valued $\mathcal{F}_t$-Wiener process W and a continuous $\mathcal{F}_t$-adapted H-valued process $X = (X_t) = (X_t)_{t\in[0,T]}$ such that, $\mathbb{P}$-a.s. the previous integral equation holds

We prove **pathwise uniqueness** (for any initial condition $x \in H$). This means that given two (weak) mild solutions (X_τ) and (X'_τ) starting at $x \in H$, we have that $X_\tau = X'_\tau$, $\mathbb{P}$-a.s., $\tau \in [0, T]$.

Theorem 1 (Masiero - P.) *Assume Hypotheses 1 and 2. For equation* (3) *pathwise uniqueness holds. Moreover, there exists* $c_T > 0$ *such that*

$$\sup_{\tau\in[0,T]} \mathbb{E}\big[\, |X_\tau^{x_1} - X_\tau^{x_2}|_H^2 \,\big] \leqslant c_T |x_1 - x_2|_H^2, \qquad x_1, x_2 \in H.$$

Remark. Our result *implies strong existence* by the Yamada-Watanabe principle (see [Ondreját, Dissertationes Math. 2004]).

Let us consider the following semilinear deterministic wave equation for $\tau \in [0, T]$:

$$\begin{cases} \frac{\partial^2 y}{\partial \tau^2}(\tau, \xi) = \frac{\partial^2 y}{\partial \xi^2}(\tau, \xi) + b(\xi, y(\tau, \xi)) \\ y(\tau, 0) = y(\tau, \pi) = 0, \\ y(0, \xi) = 0, \quad \frac{\partial y}{\partial \tau}(0, \xi) = 0, \quad \xi \in [0, \pi]. \end{cases} \tag{5}$$

with

$$b(\xi, y) = 56 \sqrt[4]{\sin \xi \, |y|^3} \cdot I_{\{|y|<2T^8\}} + |y| \cdot I_{\{|y|<2T^8\}} + 56 \sqrt[4]{8T^{24} \sin \xi} \cdot I_{\{|y|\geqslant 2T^8\}} + 2T^8 \, I_{\{|y|\geqslant 2T^8\}},$$

where $\xi \in [0, \pi]$, $y \in \mathbb{R}$; I_A is the indicator function of a set $A \subset \mathbb{R}$,

It turns out that $\boxed{y(\tau, \xi) \equiv 0}$ is a solution to equation (5), and also $\boxed{y(\tau, \xi) = \tau^8 \sin \xi}$ is a solution to (5).

• **Kolmogorov PDEs**: we find an H-valued solution v of the following equation which contains the wave operator A:

$$\boxed{\begin{cases} \dfrac{\partial v(t,x)}{\partial t} + \mathcal{L}_t[v(t,\cdot)](x) = Av(t,x) - GB(t,x),\ x \in D(A),\quad t \in [0,T], \\ \\ v(T,x) = 0. \end{cases}}$$

where for scalar and regular $f : H \to \mathbb{R}$

$$\mathcal{L}_t[f](x) = \frac{1}{2}\mathrm{Tr}\ GG^*\ \nabla^2 f(x) + \langle Ax, \nabla f(x)\rangle + < GB(t,x), \nabla f(x)\rangle.$$

• We solve the Kolmogorov equation in the form

$$v(t,x) = \int_t^T R_{s-t}\left[e^{-(s-t)A}GB(s,\cdot)\right](x)\,ds + \int_t^T R_{s-t}\left[e^{-(s-t)A}\nabla^G v(s,\cdot)B(s,\cdot)\right](x)\,ds,$$

where ∇^G gradient in the direction of $GU = \left\{\begin{pmatrix} 0 \\ a \end{pmatrix}\right\}_{a\in U}$ and R_t is the OU transition Markov semigroup associated to the linear stochastic wave equation.

We need the Banach space E_0 consisting of all $u \in B_b([0,T] \times H, H)$ such that $u(t,\cdot)$ is Fréchet differentiable on H, with Fréchet derivative $\nabla u \in B_b([0,T] \times H, L(H,H))$ and $\nabla^G u \in B_b([0,T] \times H, L_2(U,H))$. For each $\xi \in U$, $t \in [0,T]$, the map:

$$x \mapsto \nabla^G_\xi u(t,x) \text{ is Fréchet differentiable on } H \text{ with } \sup_{t,x} \sup_{|\xi|_U=1} \|\nabla \nabla^G_\xi u(t,x)\|_{L(H,H)} < \infty.$$

Regularity lemma (Masiero, P.) There exists a unique solution $v \in E_0$. Moreover, for each $x, k \in H$, $t \in [0,T]$, the map: $\xi \to \nabla_k \nabla^G_\xi v(t,x)$ belongs to $L_2(U,H)$ and, for any $k \in H$, the mapping:

$$(t,x) \mapsto \nabla_k \nabla^G_\cdot v(t,x) \quad \text{is measurable from } [0,T] \times H \text{ into } L_2(U,H) \tag{6}$$

and

$$\sup_{x \in H, t \in [0,T]} \|\nabla_k \nabla^G_\cdot v(t,x)\|_{L_2(U,H)} \leqslant c|k|, \; k \in H,$$

for some $c > 0$ (independent of k).

Finally, there exists a function $h(r) = c(r,\alpha) > 0$, $r \geqslant 0$, such that $h(r) \to 0$ as $r \to 0^+$ and if $S \in [0,T]$ verifies $h(T-S) \cdot (\sup_{t \in [0,T]} \|B(t,\cdot)\|_\alpha) \leqslant 1/4$, then

$$\sup_{t \in [S,T],\, x \in H} \|\nabla v(t,x)\|_{L(H,H)} \leqslant 1/3. \tag{7}$$

To prove the previous regularity lemma we need to investigate regularity properties of the OU semigroup.

Recall the Ornstein Uhlenbeck process for the stochastic wave equation (i.e. $B = 0$):

$$d\Xi_\tau^{0,x} = A\Xi_\tau^{0,x} d\tau + G dW_\tau, \quad \tau \in [0,T], \quad \Xi_0^{0,x} = x \in H.$$

Two OU transition semigroups:

$$P_\tau[\phi](x) = \mathbb{E}\phi\left(\Xi_\tau^{0,x}\right), \quad \phi \in B_b(H,\mathbb{R}),$$

$$R_\tau[\Phi](x) = \mathbb{E}\Phi\left(\Xi_\tau^{0,x}\right), \quad \Phi \in B_b(H,H).$$

Note that $(R_\tau)_{\tau\geqslant 0}$ is an H-valued transiton semigroup

Controllability and minimal energy

Consider the control system

$$\begin{cases} \dot{w}(t) = Aw(t) + Gu(t), \\ w(0) = k \in H, \end{cases}$$

null controllable at time $t > 0 \leftrightarrow \operatorname{Im} e^{tA} \subset \operatorname{Im} Q_t^{1/2}$ where $Q_t = \int_0^t e^{sA} G G^* e^{sA^*} ds$

- $|Q_t^{-1/2} e^{tA} h|_H \leqslant \frac{c}{t^{3/2}} |h|_H, \quad h \in H,\ t \in (0, T)$ (known result; see also [Avalos-Lasiecka, Ann. Sc. Norm. Super. Pisa 2002] for a more general result).

We prove that for

$$\begin{cases} \dot{w}(t) = Aw(t) + Gu(t), \\ w(0) = k \in \operatorname{Im}(G), \end{cases}$$

we have

$$|Q_t^{-1/2} e^{tA} G a|_H \leqslant \frac{c}{t^{1/2}} |Ga|_H = \frac{c}{t^{1/2}} |\Lambda^{-1/2} a|_U, \quad a \in U, \qquad Ga = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

Lemma (first order regularization) $\Phi \in C_b(H, H)\ \forall\, t > 0$

$$\nabla_k R_t[\Phi](x) = \nabla R_t[\Phi](x)h = \int_H \langle Q_t^{-\frac{1}{2}} e^{tA} k, Q_t^{-\frac{1}{2}} y\rangle\, \Phi(e^{tA}x + y)\, \mathcal{N}(0, Q_t)(dz),\ \ k \in H;$$

(for the proof of such formula see [Da Prato-Zabczyk 92])

$$\nabla_a^G R_t[\Phi](x) = \int_H \langle Q_t^{-\frac{1}{2}} e^{tA} G a, Q_t^{-\frac{1}{2}} y\rangle\, \Phi(e^{tA}x + y)\mathcal{N}(0, Q_t)(dz), \quad a \in U.$$

$\nabla R_t[\Phi] \in C_b(H, L(H, H))$, $\nabla^G R_t[\Phi] \in C_b(H, L(U, H)) \cap C_b(H, L_2(U, H))$

$$\sup_{x \in H} |\nabla_k R_t[\Phi](x)| \leqslant \frac{c}{t^{\frac{3}{2}}} \|\Phi\|_\infty |k|_H,;$$

$$\sup_{x \in H} |\nabla_a^G R_t[\Phi](x)| \leqslant \frac{c}{t^{\frac{1}{2}}} \|\Phi\|_\infty |\Lambda^{-1/2} a|_U. \ \ \sup_{x \in H} \|\nabla^G R_t[\Phi](x)\|_{L_2(U,H)} \leqslant \frac{c}{t^{\frac{1}{2}}} \|\Phi\|_\infty.$$

Lemma (second order regularization) $\Phi \in C_b(H,H)$, $x, k \in H$, $\xi \in U$; $\forall\, t > 0$

$$\nabla_k \nabla_\xi^G R_t[\Phi](x) = \int_H \left(\langle \Gamma_t k, Q_t^{-\frac{1}{2}} y \rangle \, \langle \Gamma_t G\xi, Q_t^{-\frac{1}{2}} y \rangle - \langle \Gamma_t k, \Gamma_t G\xi \rangle \right) \Phi(e^{tA}x + y) \mathcal{N}(0, Q_t) dz),$$

$$\sup_{x \in H} \|\nabla_k (\nabla^G R_t[\Phi])(x)\|_{L_2(U,H)} \leqslant \frac{c|k|_H}{t^2} \|\Phi\|_\infty,$$

$$\lim_{x \to 0} \sup_{|k|=1} \sup_{y \in H} \|\nabla_k(\nabla^G R_t[\Phi])(x+y) - \nabla_k(\nabla^G R_t[\Phi])(y)\|_{L_2(U,H)} = 0.$$

Lemma (interpolation) $\Phi \in C_b^\alpha(H,H)$, $k \in H$, $\forall\, t > 0$

$$\sup_{x \in H} |\nabla_k R_t[\Phi](x)|_H \leqslant \frac{c}{t^{\frac{3}{2}(1-\alpha)}} \|\Phi\|_\alpha |k|_H;$$

$$\sup_{x \in H} \|\nabla_k(\nabla^G R_t[\Phi])(x)\|_{L_2(U,H)} \leqslant \frac{c}{t^{\frac{4-3\alpha}{2}}} \|\Phi\|_\alpha |k|_H.$$

Forward-Backward system (FBSDE)

$$\begin{cases} d\Xi_\tau^{t,x} = A\Xi_\tau^{t,x} d\tau + G\, dW_\tau, & \tau \in [t,T], \\ \Xi_t^{t,x} = x, \\ -dY_\tau^{t,x} = -AY_\tau^{t,x} + GB(\tau, \Xi_\tau^{t,x})\, d\tau + Z_\tau^{t,x} B(\tau, \Xi_\tau^{t,x}) d\tau - Z_\tau^{t,x}\, dW_\tau, & \tau \in [0,T], \\ Y_T^{t,x} = 0, \end{cases}$$

FBSDE in mild formulation

$$Y_\tau^{t,x} = \int_\tau^T e^{-(s-\tau)A} GB(s, \Xi_s^{t,x})\, ds + \int_\tau^T e^{-(s-\tau)A} Z_s^{t,x} B(s, \Xi_s^{t,x})\, ds$$

$$- \int_\tau^T e^{-(s-\tau)A} Z_s^{t,x}\, dW_s, \qquad \tau \in [0,T].$$

Solution: $(Y, Z) \in L^2_{\mathcal{P}}(\Omega, C([0,T], H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T], L_2(U,H))$

(cf. Hu-Peng SAP 1991, Fuhrman-Tessitore AOP 2002, Guatteri JAMSA 2007). Here we are also using the group property of the wave equation.

Known existence and regularity results: There exists a unique solution (Y, Z) s.t.

$$\mathbb{E} \sup_{\tau \in [0,T]} |Y_\tau^{t,x}|_H^2 + \mathbb{E} \int_0^T |Z_\tau|^2_{L_2(U,H)} \leqslant C \sup_{t \in [0,T],\, x \in H} |B(t,x)|_U$$

and the H-valued map

$$v(t,x) := Y_t^{t,x} \quad \text{is deterministic}$$

If moreover $x \mapsto B(\tau, x)$, $H \to U$, Gateaux-differentiable on H, for all $\tau \in [0,T]$, then $x \mapsto (Y^{t,x}, Z^{t,x})$ is also Gateaux-differentiable on H for any t.

It follows that

$$v(t,x)$$

belong to $C([0,T] \times H, H)$, is Gâteaux differentiable with respect to x and the map

$$\nabla_x v : [0,T] \times H \to L(H,H)$$

is strongly continuous (see [Fuhrman-Tessitore 02]). Moreover

$$\boxed{Y_\tau^{t,x} = v(\tau, \Xi_\tau^{t,x}),} \quad \mathbb{P} - a.s.; \text{ for any } \tau \in [0,T] \text{ a.e.}, \boxed{Z_\tau^{t,x} = \nabla^G v(\tau, \Xi_\tau^{t,x}),} \quad \mathbb{P} - a.s.$$

By an approximation procedure on the drift B we show that

Proposition [Masiero - P.] Let $v(t,x) = Y_t^{t,x}$ as before.

Then $v \in B_b([0,T] \times H, H)$ and, for any $t \in [0,T]$, $v(t,\cdot) : H \to H$ admits the directional derivative $\nabla_{G\xi} v(t,x)$ in any $x \in H$ and along any direction $G\xi$, with $\xi \in U$.

Moreover, for any $(t,x) \in [0,T] \times H$, the map:

$$\xi \mapsto \nabla_{G\xi} v(t,x) = \nabla_\xi^G v(t,x) \in L(U,H)$$

and, for any $\xi \in U$, $\nabla_\xi^G v \in B_b([0,T] \times H, H)$ with $\sup_{(t,x)\in[0,T]\times H} \|\nabla^G v(t,x)\|_{L(U,H)} < \infty$.

Finally, for any $\tau \in [0,T]$, a.e., we have the identification in $L_2(U,H)$:

$$\nabla^G v(\tau, \Xi_\tau^{t,x}) = Z_\tau^{t,x}, \quad \mathbb{P} \text{ a.s..} \tag{8}$$

A useful representation formula for $v = Y_t^{t,x}$

By the previous result we deduce

$$
\begin{aligned}
(*) \quad v(t,x) &= \int_t^T e^{-(s-t)A} GB(s, \Xi_s^{t,x})\, ds + \int_t^T e^{-(s-t)A} Z_s^{t,x} B(s, \Xi_s^{t,x})\, ds - \int_t^T e^{-(s-t)A} Z_s^{t,x}\, dW_s \\
&= \int_t^T e^{-(s-t)A} GB(s, \Xi_s^{t,x})\, ds + \int_t^T e^{-(s-t)A} \nabla^G v(s, \Xi_s^{t,x}) B(s, \Xi_s^{t,x})\, ds \\
&- \int_t^T e^{-(s-t)A} \nabla^G v(s, \Xi_s^{t,x})\, dW_s.
\end{aligned}
$$

Formula (*) is important in the sequel. Now taking expectation

$$
\begin{aligned}
v(t,x) =& \mathbb{E} \int_t^T e^{-(s-t)A} GB(s, \Xi_s^{t,x})\, ds - \mathbb{E} \int_t^T e^{-(s-t)A} \nabla^G v(s, \Xi_s^{t,x}) B(s, \Xi_s^{t,x})\, ds \\
=& \int_t^T R_{s-t} \left[e^{-(s-t)A} GB(s, \cdot) \right] (x)\, ds - \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G v(s, \cdot) B(s, \cdot) \right] (x)\, ds
\end{aligned}
$$

and we prove that v coincides with the solution given in the previous regularity lemma. This implies that $v = Y_t^{t,x}$ has some additional regularity which we will use.

Let us come back to

$$dX_\tau^{t,x} = AX_\tau^{t,x}d\tau + GB(\tau, X_\tau^{t,x})d\tau + G\,dW_\tau, \quad \tau \in [t,T]\,, \quad X_t^{t,x} = x, \;\; \text{i.e.,}$$

$$X_\tau^{t,x} = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-s)A}GB(s, X_s^{t,x})ds + \int_t^\tau e^{(\tau-s)A}G\,dW_s, \quad \tau \in [t,T]\,, \;\; t > 0. \qquad (9)$$

We consider a new BSDE:

$$-d\widetilde{Y}_\tau^{t,x} = -A\widetilde{Y}_\tau^{t,x}d\tau + GB(\tau, X_\tau^{t,x})\,d\tau - \widetilde{Z}_\tau^{t,x}\,dW_\tau, \qquad \widetilde{Y}_T^{t,x} = 0.$$

mild formulation of the new BSDE

$$\boxed{\widetilde{Y}_\tau^{t,x} = \int_\tau^T e^{-(s-\tau)A}GB(s, X_s^{t,x})\,ds - \int_\tau^T e^{-(s-\tau)A}\widetilde{Z}_s^{t,x}\,dW_s. \qquad \tau \in [0,T].} \qquad (10)$$

We will compare (9) and (10) in order to eliminate the "bad term B"

Let us set

$$\widetilde{W}_\tau = W_\tau + \int_0^\tau B(s, X_s^{t,x}) ds, \quad \tau \in [0, T].$$

By the Girsanov theorem

$$\begin{cases} dX_\tau^{t,x} = AX_\tau^{t,x} d\tau + G d\widetilde{W}_\tau, \quad \tau \in [t, T], \\ X_\tau^{t,x} = x, \quad \tau \in [0, t], \\ \\ -d\widetilde{Y}_\tau^{t,x} = -A\widetilde{Y}_\tau^{t,x}\, d\tau + GB(\tau, X_\tau^{t,x})\, d\tau + \widetilde{Z}_\tau^{t,x} B(\tau, X_\tau^{t,x}) d\tau - \widetilde{Z}_\tau^{t,x}\, d\widetilde{W}_\tau, \qquad \tau \in [0, T], \\ \widetilde{Y}_T^{t,x} = 0, \end{cases} \tag{11}$$

$$\boxed{\widetilde{Y}_\tau^{t,x} = \int_\tau^T e^{-(s-\tau)A} GB(s, X_s^{t,x})\, ds + \int_\tau^T e^{-(s-\tau)A} \widetilde{Z}_s^{t,x}\, B(s, X_s^{t,x})\, ds - \int_\tau^T e^{-(s-\tau)A} Z_s^{t,x}\, d\widetilde{W}_s,}$$

$\tau \in [0, T]$. Note that $X^{t,x}$ is an Ornstein-Uhlenbeck process starting from x at t which is $\mathcal{F}_{t,T}^{\widetilde{W}}$-measurable (by strong uniquenes of OU)

By uniqueness for (11) we have that the law of $\widetilde{Y}^{t,x}$ is the same of $Y^{t,x}$.

In particular

$Y_t^{t,x}$ and $\widetilde{Y}_t^{t,x}$ are both deterministic and they define a unique function $v(t,x)$.

We have, for any $\tau \in [0,T]$,

$$\boxed{\widetilde{Y}_\tau^{t,x} = v(\tau, X_\tau^{t,x}),} \quad \mathbb{P} - a.s.; \tag{12}$$

$$\text{for any } \tau \in [0,T] \text{ a.e., } \boxed{\widetilde{Z}_\tau^{t,x} = \nabla^G v(\tau, X_\tau^{t,x}),} \quad \mathbb{P} - a.s.$$

Recall that the function v has the properties given in the regularity lemma.

We finally obtain

$$\boxed{v(\tau, X_\tau^{t,x}) = \int_\tau^T e^{-(s-\tau)A} GB(s, X_s^{t,x})\, ds - \int_\tau^T e^{-(s-\tau)A} \nabla^G v(\tau, X_\tau^{t,x})\, dW_s. \qquad \tau \in [0,T].}$$

$$X_\tau^x = X_\tau^{0,x} = e^{\tau A} x + \int_0^\tau e^{(\tau-s)A} GB(s, X_s^{0,x}) ds + \int_0^\tau e^{(\tau-s)A} G dW_s, \quad \tau \in [0,T].$$

Recall the link between the two BSDEs:

$$v(t,x) = Y_t^{t,x} = \widetilde{Y}_t^{t,x}$$

Proposition For $t = 0$ any mild solution of the stochastic wave equation can be rewritten as

$$X_\tau^{0,x} = e^{\tau A}x + e^{\tau A}v(0,x) - v(\tau, X_\tau^x) + \int_0^\tau e^{(\tau-s)A}\nabla^G v(s, X_s^x)\, dW_s + \int_0^\tau e^{(\tau-s)A} G\, dW_s.$$

Also with this approach the “bad” term B has been removed as in the Ito-Tanaka approach

Proof For $\tau \in [0,T]$

$$e^{-\tau A}\widetilde{Y}_\tau^{0,x} = e^{-\tau A}\int_\tau^T e^{-(s-\tau)A} GB(s, X_s^{0,x})\, ds - e^{-\tau A}\int_\tau^T e^{-(s-\tau)A}\widetilde{Z}_s^{0,x}\, dW_s.$$

for $\tau = 0$

$$\widetilde{Y}_0^{0,x} = v(0,x) = \int_0^T e^{-sA} GB(s, X_s^{0,x})\, ds - \int_0^T e^{-sA}\widetilde{Z}_s^{0,x}\, dW_s.$$

$$
\begin{aligned}
v(0,x) &= \widetilde{Y}_0^{0,x} = e^{-\tau A}\widetilde{Y}_\tau^{0,x} + \int_0^\tau e^{-sA} GB(s, X_s^x)\,ds + \int_0^\tau e^{-sA}\widetilde{Z}_s^{0,x}\,dW_s \\
&= e^{-\tau A} v(\tau, X_\tau^x) + \int_0^\tau e^{-sA} GB(s, X_s^x)\,ds + \int_0^\tau e^{-sA}\widetilde{Z}_s^{0,x}\,dW_s \\
&= e^{-\tau A} v(\tau, X_\tau^x) + \int_0^\tau e^{-sA} GB(s, X_s^x)\,ds + \int_0^\tau e^{-sA}\nabla^G v(s, X_s^x)\,dW_s
\end{aligned}
$$

So

$$
\int_0^\tau e^{(\tau-s)A} GB(s, X_s^x)\,ds = e^{\tau A} v(0,x) - v(\tau, X_\tau^x)
$$

$$
-\int_0^\tau e^{(\tau-s)A}\nabla^G v(s, X_s^x)\,dW_s
$$

$\square$

Recall

Theorem For the stochastic wave equation (3) **pathwise uniqueness** holds. $\exists\, c > 0$ s. t.

$$\sup_{\tau\in[0,T]} \mathbb{E}|X_\tau^{x_1} - X_\tau^{x_2}|_H^2 \leqslant c|x_1 - x_2|_H^2, \qquad x_1, x_2 \in H$$

Proof Let X^1, X^2 solutions starting at x_1, x_2.

We first work up to some suitable time $T_0 \leqslant T$ and solve the FBSDE up to T_0.

We know by the regularity lemma that

$$\sup_{t\in[0,T_0],\, x\in H} \|\nabla v(t,x)\|_{L(H,H)} \leqslant 1/3.$$

By the previous proposition

$$X_\tau^1 - X_\tau^2 = e^{\tau A}(x_1 - x_2) + e^{\tau A}[v(0,x_1) - v(0,x_2)]$$
$$-[v(\tau, X_\tau^1) - v(\tau, X_\tau^2)] + \int_0^\tau e^{(\tau-s)A}[\nabla^G v(s, X_s^1) - \nabla^G v(s, X_s^2)]\, dW_s.$$

By regularity properties of v

$$
\begin{aligned}
&|e^{\tau A}(x_1 - x_2)|_H + |e^{\tau A}[v(0,x_1) - v(0,x_2)]|_H + |v(\tau, X_\tau^1) - v(\tau, X_\tau^2)|_H \\
&\leqslant C|x_1 - x_2|_H + \frac{1}{3}|X_\tau^1 - X_\tau^2|_H
\end{aligned}
$$

and so

$$
|X_\tau^1 - X_\tau^2|_H \leqslant C|x_1 - x_2|_H + \frac{1}{3}|X_\tau^1 - X_\tau^2|_H
$$

$$
+ \int_0^\tau e^{(\tau - s)A}[\nabla^G v(s, X_s^1) - \nabla^G v(s, X_s^2)]\, dW_s.
$$

by Ito's isometry

$$\mathbb{E}\Big|\int_0^\tau e^{(\tau-s)A}[\nabla^G v(s,X_s^1) - \nabla^G v(s,X_s^2)]\; dW_s\Big|^2$$
$$\leqslant \mathbb{E}\int_0^\tau \|\nabla^G v(s,X_s^1) - \nabla^G v(s,X_s^2)\|^2_{L_2(U,H)}.ds$$

Using $(e_k)_k$ basis in U:

$$\mathbb{E}\int_0^\tau \|\nabla^G v(s,X_s^1) - \nabla^G v(s,X_s^2)\|^2_{L_2(U,H)} ds = \sum_{k\geqslant 1} \mathbb{E}\int_0^\tau |\nabla^G_{e_k} v(s,X_s^1) - \nabla^G_{e_k} v(s,X_s^2)|^2_H ds$$

$$\leqslant .... \leqslant \boxed{\sup_{t,x}\ \sup_{|k|_H=1} \|\nabla_k \nabla^G v(t,x)\|^2_{L_2(U,H)}} \cdot \int_0^\tau \mathbb{E}|X_s^1 - X_s^2|^2_H ds.$$

The above quantity $\boxed{...}$ is finite by the regularity lemma.

Gronwall Lemma: assertion up to time T_0.

If $T_0 < T$, continuing, solving a FBSDE on $[0,(2T_0)\wedge T]$, ... , $\rightsquigarrow$ assertion on $[0,T]$ $\square$

Backward stochastic differential equations in infinite dimensions

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Thank you

Let $D \subset \mathbb{R}^2$ be a bounded open domain with smooth boundary ∂D, which represents an elastic plate. We consider the following semilinear stochastic plate equation

$$\begin{cases} \frac{\partial^2 y}{\partial \tau^2}(\tau,\xi) = \triangle^2 y(\tau,\xi) + b(\tau,\xi,y(\tau,\xi)) + \dot{W}(\tau,\xi), \\ y(\tau,z) = 0, \quad \frac{\partial y}{\partial \nu}(\tau,z) = 0, \quad z \in \partial D, \\ y(0,\xi) = x_0(\xi), \quad \frac{\partial y}{\partial \tau}(0,\xi) = x_1(\xi), \quad \tau \in (0,T], \ \xi \in D, \end{cases} \tag{13}$$

where $\triangle$ is the Laplacian in ξ, $\triangle^2 = \triangle(\triangle)$ is a fourth order operator, $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on the boundary (we are considering the so-called clamped boundary conditions).

We introduce $U = L^2(D)$; the operator $\Lambda = \triangle^2$, with domain

$$D(\Lambda) = H^4(D) \cap H_0^2(D)$$

is a positive self-adjoint operator ($H_0^2(D)$ is the closure of $C_0^\infty(D)$ in $H^2(D)$

One can prove that $D(\Lambda^{1/2}) = H_0^2(D)$ The topological dual of $H_0^2(D)$ will be indicated by $H^{-2}(D)$.

In order to check that Λ satisfies our hypotheses recall a classical result by Courant states that the eigenvalues λ_n of Λ have the asymptotic behaviour

$$\lambda_n \sim \frac{(4\pi n)^2}{f^2} \tag{14}$$

where f denotes the area of D (such behaviour depends on the size but not on the shape of the plate).

It follows that Λ^{-1} is a trace class operator in $L^2(D)$. Proceeding as in Sections 2 and 3.1 we consider an extension of Λ to $H^{-2}(D)$ with domain $H_0^2(D)$.

The initial conditions of (13) are $x_0 \in L^2(D)$, $x_1 \in H^{-2}(D)$.

The reference Hilbert space for the solution $X_\tau(\xi) := \begin{pmatrix} y(\tau,\xi) \\ \frac{\partial}{\partial\tau} y(\tau,\xi) \end{pmatrix}$ is $H = L^2(D) \times H^{-2}(D)$.

Hypothesis 2 *The function* $b : [0,T] \times D \times \mathbb{R} \to \mathbb{R}$ *is measurable and, for* $\tau \in [0,T]$ *and a.a.* $\xi \in D$, *the map* $b(\tau,\xi,\cdot) : \mathbb{R} \to \mathbb{R}$ *is continuous. There exists* c_1 *bounded and measurable on* D*,* $\alpha \in (2/3, 1)$*, such that, for* $\tau \in [0,T]$ *and for a.a.* $\xi \in D$,

$$|b(\tau,\xi,x) - b(\tau,\xi,y)| \leqslant c_1(\xi)\, |x-y|^{\alpha},$$

$x,\ y \in \mathbb{R}$*. Moreover* $|b(\tau,\xi,x)| \leqslant c_2(\xi)$, *for* $\tau \in [0,T]$ *and a.a.* $\xi \in D$*, with* $c_2 \in L^2(D)$*.*

We consider here a positive self-adjoint operator S on a real separable Hilbert space K, i.e., $S : \mathcal{D}(S) \subset K \to K$. Note that here the compactness of S^{-1} is dispensed with. We introduce the Hilbert space

$$M = \mathcal{D}\left(S^{\frac{1}{2}}\right) \times K$$

endowed with the inner product

$$\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle_M = \left\langle S^{\frac{1}{2}} y_1, S^{\frac{1}{2}} y_2 \right\rangle_K + \langle z_1, z_2 \rangle_K .$$

We also introduce

$$\mathcal{D}(A) = \mathcal{D}(S) \times \mathcal{D}\left(S^{\frac{1}{2}}\right), \quad A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -S & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \text{ for every } \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(A). \tag{15}$$

The operator A is the generator of the contractive group e^{tA} on M

$$e^{tA} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \cos \sqrt{S}t & \frac{1}{\sqrt{S}} \sin \sqrt{S}t \\ -\sqrt{S} \sin \sqrt{S}t & \cos \sqrt{S}t \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad t \in \mathbb{R}.$$

We consider the following linear controlled system in M:

$$\begin{cases} \dot{w}(t) = Aw(t) + Gu(t), \\ w(0) = k \in M, \end{cases} \tag{16}$$

where $G : K \longrightarrow M$ is defined by $Gu = \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} u$ and the control $u \in L^2_{loc}(0, \infty; K)$. We remark that $\operatorname{Im} G = \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in M : a_1 = 0 \right\}$.

It is well known that equation (16) is null controllable for any $t > 0$ and any initial state in M, see for instance [R.F. Curtain, H.J. Zwart 1995]. This is equivalent to say that, for any $t > 0$,

$$e^{tA}(M) \subset Q_t^{1/2}(M), \quad \text{where } Q_t = \int_0^t e^{sA} G G^* e^{sA^*} ds. \tag{17}$$

(cf. Section 2). Moreover the minimal energy $\mathcal{E}_C(t, k)$ steering a general initial state $k = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ to 0 in time t behaves like $t^{-\frac{3}{2}}|k|_M$ as t goes to 0 (see e.g. [Avalos-Lasiecka ASNS 2003] for a more general result). Recall that $\mathcal{E}_C(t, k)$ is the infimum of

$$\left(\int_0^t |u(r)|_K^2 dr \right)^{1/2}$$

over all controls $u \in L^2(0,t;K)$ driving the solution w from k to 0 in time t. It can be proved that

$$\mathcal{E}_C\,(t,k) = |Q_t^{-1/2} e^{tA} k|_M$$

Hence if $\mathcal{E}_C\,(t) = \sup\limits_{|k|_M=1} \mathcal{E}_C\,(t,k)$, we know that

$$\mathcal{E}_C\,(t) \quad \text{is} \quad O(t^{-3/2}), \quad \text{as} \quad t \to 0^+. \tag{18}$$

On the other hand, we have the following estimate for the minimal energy steering an initial state $k \in \mathrm{Im}(G)$ to 0 at time t.

Theorem 3 *There exists a positive constant* C *such that, for any* $k = \begin{pmatrix} 0 \\ a \end{pmatrix} \in \mathrm{Im}(G)$,

$$|\mathcal{E}_C\,(t,k)\,| \leqslant \frac{C|k|_M}{t^{\frac{1}{2}}} = \frac{C|a|_K}{t^{\frac{1}{2}}}, \quad t > 0, \tag{19}$$

A similar result has been proved in [Masiero, Appl. Math. Optim. 2005] by a spectral approach in the case of the wave equation in $H_0^1([0,1]) \times L^2([0,1])$.

The proof of Theorem 3 is inspired by [Triggiani, Journal of Optimization Theory 1992].

It is convenient to consider the Hilbert space

$$H = K \times \mathcal{D}\left(S^{-\frac{1}{2}}\right)$$

(by $\mathcal{D}\left(S^{-\frac{1}{2}}\right)$ we mean the completion of K with respect to the norm $|S^{-1/2}\cdot|$; this is a Hilbert space; see also Section 2) endowed with the inner product

$$\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle_H = \left\langle S^{-\frac{1}{2}} z_1, S^{-\frac{1}{2}} z_2 \right\rangle_K + \langle y_1, y_2 \rangle_K \,. \tag{20}$$

We also consider an extension of the unbounded wave operator A which we still denote by A:

$$\mathcal{D}(A) = \mathcal{D}\left(S^{1/2}\right) \times K, \qquad A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -S & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -Sy \end{pmatrix} \in H, \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(A)\,.$$

Clearly A generates a contractive group e^{tA} on M and moreover if $a \in K$ we have

$$k = \begin{pmatrix} 0 \\ a \end{pmatrix} \in \mathcal{D}(A) \ \text{ and } \ e^{tA} \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{S}} \sin(\sqrt{S}\,t)\,a \\ \cos(\sqrt{S}t)a \end{pmatrix} \in D(S^{1/2}) \times K, \qquad t \in \mathbb{R}. \tag{21}$$

Let us fix $T > 0$ and $k = \begin{pmatrix} 0 \\ a \end{pmatrix}$ with $a \in K$. Consider $f(t) = t^2(T-t)^2$ and

$$\phi(t) = \frac{f(t)}{\int_0^T f(s)ds}, \quad t \in [0, T].$$

Note that $\phi(0) = \phi(T)$, $\int_0^T \phi(s)ds = 1$ and there exists $C > 0$ (independent of $T > 0$) such that $|\phi(t)| \leqslant \frac{C}{T}$ and $|\phi'(t)| \leqslant \frac{C}{T^2}$, $t \in [0, T]$. Let $\psi : [0, T] \to H$,

$$\psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = -\phi(t)\, e^{tA} k = -\begin{pmatrix} \phi(t) \frac{1}{\sqrt{S}} \sin(\sqrt{S}\, t)\, a \\ \phi(t) \cos(\sqrt{S}\, t) a \end{pmatrix}, \quad t \in [0, T].$$

Using also the derivative ψ_1' we introduce the control

$$u(t) = \psi_2(t) + \psi_1'(t) \in K, \quad t \in [0, T]. \tag{22}$$

We show that it transfers k to 0 at time T. We have

$$\int_0^T e^{(T-s)A} G u(s) ds = \int_0^T e^{(T-s)A} \begin{pmatrix} 0 \\ \psi_2(s) \end{pmatrix} ds + \int_0^T e^{(T-s)A} G \psi_1'(s) ds.$$

Since $G\psi_1'(s) = \begin{pmatrix} 0 \\ \psi_1'(s) \end{pmatrix}$ is continuous from $[0, T]$ with values in $\mathcal{D}(A)$ (cf. (21))

Integrating by parts we get

$$\int_0^T e^{(T-s)A} G\psi_1'(s) = \int_0^T e^{(T-s)A} AG\psi_1(s)ds = \int_0^T e^{(T-s)A} \begin{pmatrix} 0 & I \\ -S & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \psi_1(s) \end{pmatrix} ds$$

$$= \int_0^T e^{(T-s)A} \begin{pmatrix} \psi_1(s) \\ 0 \end{pmatrix} ds.$$

Hence we find

$$\int_0^T e^{(T-s)A} Gu(s)ds = -\int_0^T \phi(s) e^{(T-s)A} e^{sA} k\, ds = -e^{TA}k.$$

Now we compute the energy of the control u: $\int_0^T |u(s)|^2 ds$. First note that

$$\int_0^T |\psi_2(t)|_K^2 dt = \int_0^T \phi(t)^2 |\cos(\sqrt{S}\,t)a|_K^2 dt \leqslant \frac{|a|_K^2}{T}$$

On the other hand using the spectral theorem for self-adjoint operators we get

$$\int_0^T |\psi_1(t)'|_K^2 dt = \int_0^T \left| \phi(t)\cos(\sqrt{S}\,t)a + \phi'(s)\frac{1}{\sqrt{S}}\sin(\sqrt{S}\,t)\, a \right|_K^2 dt$$

$$\leqslant \frac{2|a|_K^2}{T} + 2|a|_K^2 \int_0^T \left| \phi'(t) t \frac{1}{\sqrt{S}\,t} \sin(\sqrt{S}\,t) \right|_K^2 dt \leqslant \frac{c|a|_K^2}{T},$$

where c is independent on T. Collecting the previous estimates we obtain

$$\mathcal{E}_C\,(T,k) \leqslant \left(\int_0^T |u(s)|_K^2 ds\right)^{1/2} \leqslant \frac{C\,|k|_K}{\sqrt{T}}, \quad T > 0.$$

Now let U be a real separable Hilbert space and let $\Lambda : D(\Lambda) \subset U \to U$ be a positive self-adjoint operator on U. We also consider the Hilbert space

$$V = D(\Lambda^{1/2})$$

and its dual space V' which can be identified with the completion of U with respect to the norm $|\Lambda^{-1/2} \cdot |_U$ (see the book Tucsnak and G. Weiss 09).

The operator Λ can be extended to a positive self-adjoint operator on V' which we still denote by Λ with domain V:

$$\Lambda : V \subset V' \to V'. \tag{23}$$

It turns out that the square root of such extension has domain $U \subset V'$ (the proof of this fact is simple in our case since Λ has a diagonal form; recall that we assume that Λ^{-1} is of trace class).

We need to apply Theorem 3 in the case when $K = V'$ and $S = \Lambda$. Moreover

$$M = H = U \times V',$$

$$\mathcal{D}(A) = V \times U, \quad A\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}, \text{ for every } \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(A)$$

(cf. (15)). The operator $G : V' \longrightarrow H$ is defined by $Ga = \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} a$, $a \in V'$. The associated controlled system is

$$\begin{cases} \dot{w}(t) = Aw(t) + Gu(t), \\ w(0) = h \in H, \end{cases} \tag{24}$$

with $u \in L^2_{loc}(0, \infty; V')$. By (18) and Theorem 3 we get

Corollary 4 *Let* $\mathcal{E}_C(t, h) = |Q_t^{-1/2} e^{tA} h|_H, \ \ h \in H$. *We have, for* $t \in (0, T)$

$$|Q_t^{-1/2} e^{tA} h|_H \leqslant \frac{c_T}{t^{3/2}} |h|_H$$
$$|Q_t^{-1/2} e^{tA} Ga|_H \leqslant \frac{c}{t^{1/2}} |Ga|_H = \frac{c}{t^{1/2}} |a|_{V'}, \quad h \in H, \ a \in V'.$$

In particular, since $U \subset V'$*,*

$$|Q_t^{-1/2} e^{tA} Ga|_H \leqslant \frac{c}{t^{1/2}} |Ga|_H = \frac{c}{t^{1/2}} |\Lambda^{-1/2} a|_U, \quad a \in U. \tag{25}$$